216. *A Remark on Semi-groups of Local Lipschitzians in Banach Space*

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(Comm. by Kunihiko KODAIRA, M. J. A., Sept. 13, 1971)

1. Introduction. Let X be a Banach space with the norm denoted by $\|\cdot\|$ and let C be a subset of X. A one-parameter family $\{T_t\,;\,0\leq t<+\infty\}$ of (nonlinear) operators of C into itself is called a *semi-group* on C if it satisfies the following conditions:

(i) $T_0=I|_C$ (the identity mapping restricted to C) and $T_{t+s}=T_tT_s$ for $t,\ s\geq 0$;

(ii) For each fixed $x\in C$, T_tx is strongly continuous in $t\geq 0$.

A (possibly) multiple-valued[1] operator A (with the domain $D(A)$ and the range $R(A)$) in X is said to be a *D-operator* (in the terminology of Chambers and Oharu [1]) if it satisfies the following condition:

(D) There exists a non-negative function $\omega=\omega(r)$ on $(0,+\infty)$ such that $A|_{B_r}-\omega(r)I$ is dissipative for each $r>0$; where $B_r=\{x\in X\,;\,\|x\|\leq r\}$.

The purpose of this paper is to give a sufficient condition in order that a D-operator A in X generate a semi-group on $\overline{D(A)}$ and show some examples. The condition is a modified version of that in [1].

The author wishes to express his hearty thanks to Professor S. Oharu for his stimulating suggestion.

2. Generation of semi-groups. Our theorem reads:

Theorem 1. *We assume that A is a D-operator in X and that there exists a positive function $\rho=\rho(r,T)$ on $(0,+\infty)\times(0,+\infty)$ satisfying the following condition (S_n) for each sufficiently large integer $n>2T\omega(\rho(r,T))$:*

(S_n) *The system of equations:*

$$x_\lambda^{(1)}-\lambda Ax_\lambda^{(1)}\ni x,\quad x_\lambda^{(2)}-\lambda Ax_\lambda^{(2)}\ni x_\lambda^{(1)},\ \cdots,\quad x_\lambda^{(n)}-\lambda Ax_\lambda^{(n)}\ni x_\lambda^{(n-1)},$$

has a solution $\{x_\lambda^{(1)},x_\lambda^{(2)},\cdots,x_\lambda^{(n)}\}$, where each $x_\lambda^{(\nu)}$ belongs to $B_{\rho(r,T)}\cap D(A)$, for every $x\in B_r\cap\overline{D(A)}$ and $\lambda\in(0,T/n]$.[2] *Then*

$$\text{(2,1)}\quad \exp(tA)\cdot x=\lim_{n\to\infty}\{I-(t/n)A|_{B_{\rho(r,T)}}\}^{-n}x,\quad 0\leq t\leq T,\quad x\in B_r\cap\overline{D(A)},$$

exists in X for each $r,\ T>0$, and $\{\exp(tA)\,;\,0\leq t<+\infty\}$ is a semi-group

1) For the notion of "multiple-valued" operator, we refer to Kato [5], §2.

2) Since $0<\lambda<\omega(\rho(r,T))^{-1}$, one can write that $x_\lambda^{(1)}=(I-\lambda A|_{B_{\rho(r,T)}})^{-1}x$, $x_\lambda^{(2)}=(I-\lambda A|_{B_{\rho(r,T)}})^{-2}x,\cdots,x_\lambda^{(n)}=(I-\lambda A|_{B_{\rho(r,T)}})^{-n}x$.

on $\overline{D(A)}$. *Moreover the convergence in* (2,1) *is uniform in* $t \in [0, T]$, $\exp(tA)$ *is a locally Lipschitz continuous operator for each fixed* $t \geq 0$:

$$\|\exp(tA)\cdot x - \exp(tA)\cdot y\| \leq \exp[\omega(\rho(r, T))t]\cdot\|x-y\|, \quad 0 \leq t \leq T,$$
$$x, y \in B_r \cap \overline{D(A)}, \text{ for each } T, r > 0,$$

and $\exp(tA)\cdot x$ *is a locally Lipschitz continuous function of* $t \geq 0$ *for each fixed* $x \in D(A)$.

Sketch of the proof. We set $T, r > 0$, $n \geq m \gg 2T\omega(\rho(r, T))$, $\mu \in (0, T/n]$, $\lambda \in (0, T/m]$, $\mu \leq \lambda$ and $x \in B_r \cap D(A)$. Then by the argument parallel to that used in the proof of Theorem I in Crandall-Liggett [2], we obtain immediately the following estimate:

$$\begin{aligned}&\|(I-\mu A|_{B_{\rho(r,T)}})^{-n}x - (I-\lambda A|_{B_{\rho(r,T)}})^{-m}x\| \\ &\quad\leq \{[(n\mu-\lambda m)^2 + n\mu(\lambda-\mu)]^{1/2}\cdot\exp[2\omega(\rho(r, T))(n\mu+m\lambda)] \\ &\quad\quad + [m\lambda(\lambda-\mu) + (m\lambda - n\mu)^2]^{1/2}\cdot\exp[4\omega(\rho(r, T))n\mu]\}\cdot|||Ax|||;\end{aligned}$$

where $|||Ax||| = \inf\{\|y\|; y \in Ax\}$ (cf. (1.9) of [2]). This surely enables us to prove Theorem 1.

3. Examples. Let S be a uniform space and $(C(S), \|\cdot\|_\infty)$ be the Banach lattice of all real-valued bounded uniformly continuous functions on S normed with the supremum of the absolute value. We denote by $C(S)^+$ the cone of all non-negative elements of $C(S)$ and define the norm $\|\cdot\|$ in the Banach space $X = C(S)\times C(S)$ by the relation:

$$\|(u, v)\|^2 = \|u\|_\infty^2 + \|v\|_\infty^2, \qquad (u, v) \in X.$$

Suppose we are given a generator Λ of a "sub-Markov semi-group"[3] (of contractions) on $C(S)$: Λ is a densely defined closed linear operator such that $R(I-\lambda\Lambda) = C(S)$ for any $\lambda > 0$ and

$$\sigma((f-c)^+, \Lambda f) \leq 0 \tag{3,1}$$

holds for every $f \in D(\Lambda)$ and every non-negative constant function c on S where

$$\sigma(f, g) = \begin{cases} \lim\limits_{\varepsilon\downarrow 0} \sup\limits_{s\in S(f,\varepsilon)} g(s), & f \neq 0,\ f \geq 0, \\ 0, & f = 0; \end{cases} \tag{3,2}$$

$S(f, \varepsilon) = \{s \in S; |f(s)| > \|f\|_\infty - \varepsilon\}$ (see Sato [7]), in other words, Λ is "completely dispersive" in Kunita's terminology. We define two operators A_1 and A_2 in X by the relations:

$$\begin{aligned} &D(A_i) = (D(\Lambda)\cap C(S)^+)\times C(S)^+ \subset X, && i = 1, 2; \\ &A_1(u, v) = (\Lambda u + uv, -uv), && (u, v) \in D(A_1), \\ &A_2(u, v) = (\Lambda u - uv, uv), && (u, v) \in D(A_2). \end{aligned} \tag{3,3}$$

Lemma 2. *Both* A_1 *and* A_2 *are D-operators in* $X = C(S)\times C(S)$ *with* $\omega(r) = 2r$ *and satisfy the assumption in Theorem* 1 *with* $\rho(r, T) = r\exp(2rT)$.

Proof. For a "semi-scalar product"[4] $[\ ,\ ]_\infty$ in $C(S)$, we define a semi-scalar product $[\ ,\ ]$ in $C(S)\times C(S)$ by the relation:

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3) For this notion, we refer to Kunita [6].

4) For the definition, see, for example, Yosida [10].

$$[(u_1, v_1), (u_2, v_2)] = [u_1, u_2]_\infty + [v_1, v_2]_\infty, \qquad (u_i, v_i) \in X,\ i=1, 2.$$

Then we have easily that

$$[A_i(u_1, v_1) - A_i(u_2, v_2), (u_1, v_1) - (u_2, v_2)] \leq 2r\, \|(u_1, v_1) - (u_2, v_2)\|^2,$$

$i=1, 2$, for any $(u_j, v_j) \in B_r \cap D(A_i)$ $(r>0)$, $j=1, 2$. This implies that A_1 and A_2 are D-operators with $\omega(r)=2r$.

Next we shall prove (S_n), $n>2T\omega(\rho(r, T))$, for $A=A_1$ with $\rho(r, T)=r\exp(2rT)$. For each $(u, v) \in B_r \cap \overline{D(A_1)}$ and $\lambda \in (0, T/n]$, there exist solutions $u_\lambda^{(1)} \in D(\Lambda) \cap C(S)^+$, $v_\lambda^{(1)} \in C(S)^+$ of the equations: $u_\lambda^{(1)} - \lambda \Lambda u_\lambda^{(1)} - \lambda v u_\lambda^{(1)}(1+\lambda u_\lambda^{(1)})^{-1} = u$, $v_\lambda^{(1)} = v(1+\lambda u_\lambda^{(1)})^{-1}$, since $(I-\lambda\Lambda)^{-1}$ is non-negative and the operator $F_\lambda^{(1)}$ of $C(S)^+$ into itself defined by $F_\lambda^{(1)} f = vf(1+\lambda f)^{-1}$, $f \in C(S)^+$, is a Lipschitz continuous operator with the Lipschitz constant $\leq \|v\|_\infty \leq r$. Moreover we have the estimates: $\|u_\lambda^{(1)}\|_\infty \leq (1-\lambda\|v\|_\infty)^{-1}u$, $\|v_\lambda^{(1)}\|_\infty \leq \|v\|_\infty$. Accordingly the equation: $(u_\lambda^{(1)}, v_\lambda^{(1)}) - \lambda A_1(u_\lambda^{(1)}, v_\lambda^{(1)}) = (u, v)$, has a solution $(u_\lambda^{(1)}, v_\lambda^{(1)}) \in B_{r\exp(2rT)} \cap D(A_1)$. By the same argument the system of equations: $(u_\lambda^{(2)}, v_\lambda^{(2)}) - \lambda A_1(u_\lambda^{(2)}, v_\lambda^{(2)}) = (u_\lambda^{(1)}, v_\lambda^{(1)}), \cdots, (u_\lambda^{(n)}, v_\lambda^{(n)}) - \lambda A_1(u_\lambda^{(n)}, v_\lambda^{(n)}) = (u_\lambda^{(n-1)}, v_\lambda^{(n-1)})$, has a solution $(u_\lambda^{(2)}, v_\lambda^{(2)}), \cdots, (u_\lambda^{(n)}, v_\lambda^{(n)}) \in B_{r\exp(2rT)} \cap D(A_1)$, which satisfies the estimate:

$$\|u_\lambda^{(n)}\|_\infty \leq (1-\lambda\|v\|_\infty)^{-n}\|u\|_\infty, \qquad \|v_\lambda^{(n)}\|_\infty \leq \|v\|_\infty. \tag{3,4}$$

Finally we shall prove (S_n), $n>2T\omega(\rho(r, T))$, for $A=A_2$ with $\rho(r, T)=r\exp(2rT)$. For each $(u, v) \in B_r \cap \overline{D(A_2)}$ and $\lambda \in (0, T/n]$, there exists a solution $u_\lambda^{(1)} \in D(\Lambda) \cap C(S)^+$ of the equation: $u_\lambda^{(1)} - \lambda\Lambda u_\lambda^{(1)} + \lambda v u_\lambda^{(1)}(1-\lambda \min(r, u_\lambda^{(1)}))^{-1} = u$, since the operator $G_\lambda^{(1)}$ on $C(S)^+$ defined by $G_\lambda^{(1)} f = -vf(1-\lambda\min(r, f))^{-1}$ is dissipative[5] and locally Lipschitz continuous and, accordingly, one can use Theorem 1.2.2 in Da Prato [3]. In view of (3,1) and (3,2) we have (see Sato [7])

$$\begin{aligned}\|(u_\lambda^{(1)} - \|u\|_\infty)^+\|_\infty &= \sigma((u_\lambda^{(1)} - \|u\|_\infty)^+, u_\lambda^{(1)} - \|u\|_\infty) = \sigma((u_\lambda^{(1)} - \|u\|_\infty)^+,\\ &\quad u + \lambda\Lambda u_\lambda^{(1)} - \lambda v u_\lambda^{(1)}(1-\lambda\min(r, u_\lambda^{(1)}))^{-1} - \|u\|_\infty) \leq \sigma((u_\lambda^{(1)} - \|u\|_\infty)^+, u - \|u\|_\infty)\\ &\quad + \lambda\sigma((u_\lambda^{(1)} - \|u\|_\infty)^+, -v u_\lambda^{(1)}(1-\lambda\min(r, u_\lambda^{(1)}))^{-1}) \leq \|(u-\|u\|_\infty)^+\|_\infty = 0\end{aligned}$$

(see also Sato [8]). Hence $u_\lambda^{(1)} \leq \|u\|_\infty \leq r$. Accordingly $(u_\lambda^{(1)}, v_\lambda^{(1)})$, $v_\lambda^{(1)} = v(1-\lambda u_\lambda^{(1)})^{-1}$, satisfies $(u_\lambda^{(1)}, v_\lambda^{(1)}) - \lambda A_2(u_\lambda^{(1)}, v_\lambda^{(1)}) = (u, v)$, $(u_\lambda^{(1)}, v_\lambda^{(1)}) \in B_{r\exp(2rT)} \cap D(A_2)$. By the same argument the system of equations: $(u_\lambda^{(2)}, v_\lambda^{(2)}) - \lambda A_2(u_\lambda^{(2)}, v_\lambda^{(2)}) = (u_\lambda^{(1)}, v_\lambda^{(1)}), \cdots, (u_\lambda^{(n)}, v_\lambda^{(n)}) - \lambda A_2(u_\lambda^{(n)}, v_\lambda^{(n)}) = (u_\lambda^{(n-1)}, v_\lambda^{(n-1)})$, has a solution $(u_\lambda^{(2)}, v_\lambda^{(2)}), \cdots, (u_\lambda^{(n)}, v_\lambda^{(n)}) \in B_{r\exp(2rT)} \cap D(A_2)$, which satisfies the estimate:

$$\|u_\lambda^{(n)}\|_\infty \leq \|u\|_\infty, \qquad \|v_\lambda^{(n)}\|_\infty \leq (1-\lambda\|u\|_\infty)^{-n}\|v\|_\infty. \tag{3,5}$$

Q.E.D.

Applying Theorem 1 to the operators A_1 and A_2, we have by the above-mentioned Lemma 2 and in view of the estimates (3,4) and (3,5):

The operators A_1 *and* A_2 *defined by* (3,3) *generate semi-groups* $\{\exp(tA_i);\ 0 \leq t < +\infty\}$, $i=1, 2$, *respectively on the cone* $C(S)^+ \times C(S)^+$

5) This follows from the fact: $\tau(f-g, G_\lambda^{(1)} f - G_\lambda^{(1)} g) \leq 0$ whenever $f, g \in C(S)^+$; where $\tau(f, g) \equiv \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(\|f+\varepsilon g\|_\infty - \|f\|_\infty) = \lim_{\varepsilon\downarrow 0} \sup_{s\in S(f,\varepsilon)} (\operatorname{sgn} f(s))g(s)$, if $f \neq 0$, $= \|g\|_\infty$, if $f=0$ (cf. Hasegawa [4] and Sato [7]).

of $X=C(S)\times C(S)$ *in the sense of Theorem* 1. Moreover we have the following estimates: Set $(u_1(t), v_1(t))=\exp(tA_1)\cdot(u, v)$ and $(u_2(t), v_2(t))=\exp(tA_2)\cdot(u, v)$ for $u, v\in C(S)^+$ and $t\geq 0$, then

$$\|u_1(t)\|_\infty\leq \exp(t\|v\|_\infty)\cdot\|u\|_\infty, \qquad \|v_1(t)\|_\infty\leq\|v\|_\infty;$$
$$\|u_2(t)\|_\infty\leq\|u\|_\infty, \qquad \|v_2(t)\|_\infty\leq \exp(t\|u\|_\infty)\cdot\|v\|_\infty.$$

Remark. The forms of operators A_1 and A_2 are suggested by the forth and fifth examples of weakly coupled diffusion systems in Yamaguti, Kametaka and Mimura [9]:

$$\begin{cases}\partial u/\partial t=\Delta u+uv\\ \partial v/\partial t=-uv,\end{cases} \qquad \begin{cases}\partial u/\partial t=\Delta u-uv\\ \partial v/\partial t=uv.\end{cases}$$

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