215. *Integration of Equations of Imschenetsky Type by Integrable Systems*

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1. Introduction. Generalizing the method of integration due to Monge, the author gave a method of integration by integrable systems in [1] and [2]. Here we shall prove the following

Theorem. *Transform an equation of Imschenetsky type by one of the associated Imschenetsky transformations. Then the transformed equation is solved by integrable systems of order* $n-1$, *if and only if the original equation is solved by integrable systems of order* n.

This is a generalization of results in [1], [2] obtained for the Laplace transformation associated with a linear hyperbolic equation, and for the Imschenetsky transformation associated with an equation of Laplace type. In the second case the theorem was proved only for $n=1, 2$. In both the cases the author obtained the invariants of the equation whose vanishing is a necessary and sufficient condition in order that the equation may be solved by integrable systems of order n, and proved that the invariants for the original equation to be solved by integrable systems of order n are transformed to those for the transformed equation to be solved by integrable systems of order $n-1$. Here we shall prove the theorem directly, without obtaining the invariants of the equations.

2. Integrable systems of order *n*. Let us try to solve the Cauchy problem of an equation of type

$$s+f(x, y, z, p, q)=0, \tag{1}$$

integrating ordinary differential equations, in the space of $(x, y, z, p, q_1, \cdots, q_n)$. Here, $p=\partial z/\partial x$, $q=\partial z/\partial y$, $s=\partial^2 z/\partial x\partial y$, and $q_i=\partial^i z/\partial y^i$ $(q_1=q)$. The Cauchy problem in this space involving the derivatives of higher order is to find a two-dimensional submanifold which satisfies

$$\begin{aligned} dz-pdx-qdy=dq_1+f_0dx-q_2dy=dq_2+f_1dx-q_3dy \\ =\cdots=dq_{n-1}+f_{n-2}dx-q_ndy=0, \end{aligned} \tag{2}$$

and contains a given initial curve satisfying (2). Here, f_i is a function of $(x, y, z, p, q_1, \cdots, q_{i+1})$ defined inductively by

$$f_i=\left(G_i-f\frac{\partial}{\partial p}\right)f_{i-1} \quad (i\geqq 1), \quad f_0=f$$

with

$$G_i=\frac{d}{dy}+\sum_{j=1}^{i} q_{j+1}\frac{\partial}{\partial q_j} \quad (i\geqq 1), \quad \frac{d}{dy}=\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}.$$

A system of ordinary differential equations

$$(3) \qquad \frac{dx}{0}=\frac{dy}{1}=\frac{dz}{q_1}=\frac{dp}{-f}=\frac{dq_1}{q_2}=\cdots=\frac{dq_{n-1}}{q_n}=\frac{dq_n}{u}$$

with a function u of $(x, y, z, p, q_1, \cdots, q_n)$ is called an integrable system of order n, if u is a solution of the following system of two equations

$$(4) \qquad \frac{\partial u}{\partial p}=0, \quad \left(\frac{d}{dx}-\sum_{i=1}^{n} f_{i-1}\frac{\partial}{\partial q_i}\right)u+\frac{\partial f}{\partial q}u+\left(G_{n-1}-f\frac{\partial}{\partial p}\right)f_{n-1}=0,$$

where $d/dx=\partial/\partial x+p\partial/\partial z$. Suppose that along the initial curve satisfying (2) we have $dq_n+f_{n-1}dx-udy=0$. Then the surface obtained by integrating (3) under the given initial condition satisfies (2) and $dq_n+f_{n-1}dx-udy=0$, if the system (3) is integrable. Hence, in this case, the surface thus obtained gives a solution of the Cauchy problem.

3. Integrability condition. Any system of linear partial differential equations of the first order with one unknown function can be prolonged either to a complete system or to an incompatible system by adding compatibility conditions. If we get a complete system consisting of r independent equations by this prolongation, the original system is said to have the rank $m-r$, where m is the number of the independent variables. The equation (1) is said to be solved by integrable systems of order n, if the system (4) has the rank greater than zero. In this case the Cauchy problem is solved by integrating an integrable system of order n for any initial curve satisfying (2), since we can find such a solution u of (4) that satisfies $dq_n+f_{n-1}dx-udy=0$ along the given initial curve.

Suppose that the equation (1) is linear with respect to p;

$$(5) \qquad s+M(x, y, z, q)p+N(x, y, z, q)=0.$$

Then the second equation of (4) is written in the form

$$p\left(Z_n u+\frac{\partial M}{\partial q}u+M_n\right)+X_n u+\frac{\partial N}{\partial q}u+N_n=0,$$

where

$$Z_n=\frac{\partial}{\partial z}-\sum_{i=1}^{n} A_{i-1}\frac{\partial}{\partial q_i}, \qquad M_n=G_{n-1}A_{n-1}-MA_{n-1},$$

$$X_n=\frac{\partial}{\partial x}-\sum_{i=1}^{n} B_{i-1}\frac{\partial}{\partial q_i}, \qquad N_n=G_{n-1}B_{n-1}-NA_{n-1},$$

and A_i, B_i $(i\geqq 0)$ are defined inductively by

$$A_i=G_iA_{i-1}-MA_{i-1}, \quad B_i=G_iB_{i-1}-NA_{i-1}, \quad A_0=M, \quad B_0=N.$$

The functions M_n, N_n, A_i, B_i $(i\geqq 0)$ are independent of p. Hence, the equation (5) is solved by integrable systems of order n, if and only if the system of two equations

$$(6) \qquad Z_n u + \frac{\partial M}{\partial q} u + M_n = 0, \qquad X_n u + \frac{\partial N}{\partial q} u + N_n = 0$$

with independent variables $(x, y, z, q_1, \cdots, q_n)$ has the rank greater than zero.

4. Imschenetsky transformation. A set of four relations

$$(7) \qquad x' = x, \quad y' = y, \quad z' = h(x, y, z, q), \quad p' = k(x, y, z, q)$$

between $(x, y, z, p, q; x', y', z', p', q')$ is called an Imschenetsky transformation, if it satisfies

$$\frac{\partial h}{\partial q} \neq 0, \quad \frac{\partial(h, k)}{\partial(z, q)} \neq 0.$$

It gives the following transformation between the two equations from

$$(8) \qquad \frac{\partial h}{\partial q} s + \frac{\partial h}{\partial z} p + \frac{\partial h}{\partial x} - k = 0$$

to

$$(9) \qquad \frac{\partial h}{\partial q} s' - \frac{\partial k}{\partial q} q' - \frac{dk}{dy} \frac{\partial h}{\partial q} + \frac{dh}{dy} \frac{\partial k}{\partial q} = 0,$$

where we replace x, y, z, q in the second equation by

$$x = x', \quad y = y', \quad z = h'(x', y', z', p'), \quad q = k'(x', y', z', p'),$$

solving (7) with respect to x, y, z, q. Take a solution $z = \phi(x, y)$ of (8). Then the surface $z' = h(x', y', \phi(x', y'), \phi_y(x', y'))$ gives a solution of (9). Conversely take a solution $z' = \phi'(x', y')$ of (9). Then the surface $z = h'(x, y, \phi'(x, y), \phi'_{x'}(x, y))$ gives a solution of (8). These two transformations are the inverse of each other.

The original equation (8) is linear with respect to p. An equation of type (5) can be the original equation of an Imschenetsky transformation, if and only if its first invariant

$$X_1 M - Z_1 N = \frac{\partial M}{\partial x} - N \frac{\partial M}{\partial q} - \frac{\partial N}{\partial z} + M \frac{\partial N}{\partial q}$$

does not vanish. In this case the equation (5) is called an equation of Imschenetsky type.

5. Proof of Theorem. Suppose that an equation of Imschenetsky type (5) is transformed to an equation

$$s' + f'(x', y', z', p', q') = 0$$

by the Imschenetsky transformation (7). Then we have

$$Z_1 h = 0, \quad X_1 h = k, \quad -f' = \frac{dk}{dy} + \left(\frac{\partial h}{\partial q}\right)^{-1} \left(q' - \frac{dh}{dy}\right) \frac{\partial k}{\partial q}.$$

Let us transform the independent variables $(x, y, z, q_1, \cdots, q_n)$ and the unknown function u of the system (6) to $(x', y', z', p', q'_1, \cdots, q'_{n-1})$ and u' respectively by

$$x' = x, \quad y' = y, \quad z' = h, \quad p' = k, \quad q' = G_1 h = \frac{dh}{dy} + \frac{\partial h}{\partial q} q_2,$$

$$q_i' = G_i G_{i-1} \cdots G_1 h = G_{i-1}(G_{i-1} \cdots G_1 h) + \frac{\partial h}{\partial q} q_{i+1}, \qquad 1 < i < n$$

and

$$u' = G_{n-1}(G_{n-1} \cdots G_1 h) + \frac{\partial h}{\partial q} u.$$

Then the first and the second equations of (6) are expressed in the forms

$$-\left(\frac{\partial h}{\partial q}\right)^{-2} \frac{\partial(h, k)}{\partial(z, q)} \frac{\partial u'}{\partial p'} = 0$$

and

$$\left(\frac{\partial h}{\partial q}\right)^{-1} \left\{ \frac{du'}{dx'} - \sum_{i=1}^{n-1} f_{i-1}' \frac{\partial u'}{\partial q_i'} + \frac{\partial f'}{\partial q'} u' + \left(G_{n-2}' - f' \frac{\partial}{\partial p'}\right) f_{n-2}' \right\} + \left(\frac{\partial h}{\partial q}\right)^{-1} X_1 k \frac{\partial u'}{\partial p'} = 0$$

respectively, where

$$G_i' = \frac{d}{dy'} + \sum_{j=1}^{i} q_{j+1}' \frac{\partial}{\partial q_j'}, \quad f_i' = \left(G_i' - f' \frac{\partial}{\partial p'}\right) f_{i-1}', \quad f_0' = f'.$$

It can be proved by

$$Z_n z' = Z_n h = 0, \quad X_n z' = X_n h = k, \quad Z_n p' = Z_n k = -\left(\frac{\partial h}{\partial q}\right)^{-1} \frac{\partial(h, k)}{\partial(z, q)},$$

$$Z_n q_i' = Z_n (G_i \cdots G_1) h = 0, \quad X_n q_i' = X_n (G_i \cdots G_1) h = -f_{i-1}', \quad 1 \leqq i < n,$$

and

$$Z_n \left(\frac{\partial h}{\partial q}\right)^{-1} = -\frac{\partial M}{\partial q} \left(\frac{\partial h}{\partial q}\right)^{-1}, \quad X_n \left(\frac{\partial h}{\partial q}\right)^{-1} = -\left(\frac{\partial N}{\partial q} - \frac{\partial f'}{\partial q'}\right) \left(\frac{\partial h}{\partial q}\right)^{-1},$$

$$Z_n G_{n-1}(G_{n-1} \cdots G_1) h = M_n \frac{\partial h}{\partial q},$$

$$X_n G_{n-1}(G_{n-1} \cdots G_1) h = N_n \frac{\partial h}{\partial q} - \left(G_{n-2}' - f' \frac{\partial}{\partial p'}\right) f_{n-2}' - \frac{\partial f'}{\partial q'} G_{n-1}(G_{n-1} \cdots G_1) h.$$

These identities can be obtained by

$$[Z_{i+1}, G_i] = -M Z_i + G_i A_i \frac{\partial}{\partial q_{i+1}}, \qquad i \geqq 1,$$

$$[X_{i+1}, G_i] = -N Z_i + G_i B_i \frac{\partial}{\partial q_{i+1}}, \qquad i \geqq 1,$$

$$G_i = G_{i-1}' - f' \frac{\partial}{\partial p'} + \sum_{j=i}^{n-1} G_i (G_j \cdots G_1) h \frac{\partial}{\partial q_j'}, \quad i \geqq 1 \quad \left(G_0' = \frac{d}{dy'}\right),$$

where

$$Z_i = \frac{\partial}{\partial z} - \sum_{j=1}^{i} A_{j-1} \frac{\partial}{\partial q_j}, \quad X_i = \frac{\partial}{\partial x} - \sum_{j=1}^{i} B_{j-1} \frac{\partial}{\partial q_j}, \quad i \geqq 1.$$

Hence, the system (6) with independent variables $(x, y, z, q_1, \cdots, q_n)$ and the system of two equations

$$\frac{\partial u'}{\partial p'}=\frac{du'}{dx'}-\sum_{i=1}^{n-1}f'_{i-1}\frac{\partial u'}{\partial q'_i}+\frac{\partial f'}{\partial q'}u'+\left(G'_{n-2}-f'\frac{\partial}{\partial p'}\right)f'_{n-2}=0$$

with independent variables $(x', y', z', p', q'_1, \cdots, q'_{n-1})$ have the same rank.

References

[1] M. Matsuda: Two methods of integrating Monge-Ampère's equations. Trans. Amer. Math. Soc., **150**, 327–343 (1970).

[2] ——: Two methods of integrating Monge-Ampère's equations. II (to appear in Trans. Amer. Math. Soc., **167** (1972)).