

207. An Application of A. Robinson's Proof of the Completeness Theorem

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In [1], A. Robinson gave a proof of the Gödel's Completeness Theorem. Robinson's argument seems to be applicable in other branches. In this note, we shall give an application in universal algebra. There is a theorem, called Grätzer's theorem, which argues a kind of finiteness property with respect to the existence of homomorphisms (cf. [2], p. 138). The usual proof of this theorem which uses the condition for the inverse limit to be non-void is rather complicated. We shall give a simplified direct proof of the extended version of this theorem, which is a slight modification of Robinson's argument.

In the following, we shall use the usual set-theoretical notation. Thus, a mapping $\varphi: A \rightarrow B$ is a subset of $A \times B$ satisfying certain conditions. For a mapping φ , $\text{Dom}(\varphi)$ denotes the domain of φ and $\text{Ra}(\varphi)$ the range of φ . If C is a set, then $\varphi|C$ denotes the restriction of φ to $C \cap \text{Dom}(\varphi)$.

A structure $\mathfrak{A} = \langle A, R_j^{\mathfrak{A}} \rangle_{j \in J}$ is a set A together with an indexed set $\{R_j^{\mathfrak{A}}\}_{j \in J}$ of finitary operations and relations on A . Then A is the universe of \mathfrak{A} . In the following, we shall consider only the relational systems, that is the structure with no operation. This restriction entails no loss of generality, because the structures can be considered as the relational systems by replacing the operations by their graphs. Then, our version of the Grätzer's theorem is the following:

Let $\mathfrak{A} = \langle A, R_j^{\mathfrak{A}} \rangle_{j \in J}$ be a finite relational system. A relational system $\mathfrak{B} = \langle B, R_j^{\mathfrak{B}} \rangle_{j \in J}$ has a homomorphism into \mathfrak{A} if and only if every finite subsystem of \mathfrak{B} has a homomorphism into \mathfrak{A} .

The "only if" part is trivial. To prove "if" part, assume that every finite subsystem of \mathfrak{B} has a homomorphism into \mathfrak{A} , and let Σ be the set of all homomorphisms of finite subsystems of \mathfrak{B} into \mathfrak{A} . Now, let Ω be the set of mappings φ from subsets of B into A satisfying the following condition:

$$(*) \left\{ \begin{array}{l} \text{For every finite subset } C \text{ of } B, \text{ there is a} \\ \text{homomorphism } \psi \text{ in } \Sigma \text{ such that the domain} \\ \text{of } \psi \text{ is } C \text{ and } \varphi|C = \psi| \text{Dom}(\varphi). \end{array} \right.$$

Since every finite subsystem of \mathfrak{B} has a homomorphism into \mathfrak{A} , the

empty mapping belongs to Ω , and hence Ω is not empty. Now, we shall introduce the partial ordering $<$ in Ω by saying that $\varphi < \varphi'$ if and only if $\text{Dom}(\varphi) \subset \text{Dom}(\varphi')$ and $\varphi = \varphi' \upharpoonright \text{Dom}(\varphi)$. Then, the ordered set Ω with $<$ is an inductively ordered set. To show this, let $\Gamma = \{\varphi_i : i \in I\}$ be a chain in Ω with respect to $<$, and put $\varphi = \cup\{\varphi_i : i \in I\}$. Obviously, φ is a mapping. Let C be a finite subset of B . Since $\text{Dom}(\varphi) = \cup\{\text{Dom}(\varphi_i) : i \in I\}$ and $\{\text{Dom}(\varphi_i) : i \in I\}$ is linearly ordered by the set-theoretical inclusion \subset , there is an $i \in I$ such that $C \cap \text{Dom}(\varphi) \subset \text{Dom}(\varphi_i)$, that is $\varphi \upharpoonright C = \varphi_i \upharpoonright C$. For the φ_i , by the definition of Ω , there is a ψ in Σ such that the domain of ψ is C and $\psi \upharpoonright \text{Dom}(\varphi_i) = \varphi_i \upharpoonright C$, and since, for the ψ , $\psi \upharpoonright \text{Dom}(\varphi_i) = \psi \upharpoonright C \cap \text{Dom}(\varphi_i) = \psi \upharpoonright \text{Dom}(\varphi)$, $\varphi \upharpoonright C = \psi \upharpoonright \text{Dom}(\varphi)$. Hence, φ is in Ω . So, Ω with $<$ is an inductively ordered set, and, by the Zorn's lemma, Ω has a maximal element $\bar{\varphi}$ with respect to $<$. We shall show that $\bar{\varphi}$ is a homomorphism from \mathfrak{B} into \mathfrak{A} .

We first show that $\text{Dom}(\bar{\varphi}) = B$. Suppose the contrary, and assume that $a \in B - \text{Dom}(\bar{\varphi})$. Let $A = \{a_1, \dots, a_n\}$ and define the mappings $\bar{\varphi}_i, i = 1, \dots, n$, with the domains $\text{Dom}(\bar{\varphi}) \cup \{a\}$ by the following:

$$\begin{aligned} \bar{\varphi}_i(x) &= \bar{\varphi}(x) & \text{for } x \in \text{Dom}(\bar{\varphi}), \\ \bar{\varphi}_i(a) &= a_i. \end{aligned}$$

By the maximality of $\bar{\varphi}$ in Ω with respect to $<$, $\bar{\varphi}_i$ is not in Ω for $i = 1, \dots, n$. Hence, for $i = 1, \dots, n$, there exists a finite subset C_i of B such that, for every ψ in Σ , $\text{Dom}(\psi) = C_i$ implies $\psi \upharpoonright \text{Dom}(\bar{\varphi}_i) \neq \bar{\varphi}_i \upharpoonright C_i$. Now, let C be a finite subset of B such that $a \notin C$. Since $\bar{\varphi}$ is in Ω , there exists a ψ in Σ with the domain C such that $\bar{\varphi} \upharpoonright C = \psi \upharpoonright \text{Dom}(\bar{\varphi})$, and since $\bar{\varphi} \upharpoonright C = \bar{\varphi}_i \upharpoonright C$ and $\psi \upharpoonright \text{Dom}(\bar{\varphi}) = \psi \upharpoonright \text{Dom}(\bar{\varphi}_i)$, $i = 1, \dots, n$, $\bar{\varphi}_i \upharpoonright C = \psi \upharpoonright \text{Dom}(\bar{\varphi}_i)$, $i = 1, \dots, n$. So, $a \notin C$ implies $C_i \neq C$, $i = 1, \dots, n$. Therefore, $a \in C_1 \cap \dots \cap C_n$. Let $\bar{C} = C_1 \cup \dots \cup C_n$. Since \bar{C} is finite, there exists a ψ in Σ with the domain \bar{C} such that $\bar{\varphi} \upharpoonright \bar{C} = \psi \upharpoonright \text{Dom}(\bar{\varphi})$. Take such a ψ_0 , and assume $\psi_0(a) = a_i$. Let $\bar{\psi}_0 = \psi_0 \upharpoonright C_i$. Then, obviously $\bar{\psi}_0$ is in Σ with $\text{Dom}(\bar{\psi}_0) = C_i$ and $\bar{\psi}_0 \upharpoonright \text{Dom}(\bar{\varphi}_i) = \bar{\varphi}_i \upharpoonright C_i$, which contradicts the above choice of C_i . Thus, we obtain a desired contradiction from the assumption $\text{Dom}(\bar{\varphi}) \neq B$. Therefore, the domain of $\bar{\varphi}$ is B .

Finally, to prove that $\bar{\varphi}$ is a homomorphism, we must show that, for every $R_i^{\mathfrak{B}}$ with n argument places, $i \in I$, and for every b_1, \dots, b_n in B , if $R_i^{\mathfrak{B}}(b_1, \dots, b_n)$, then $R_i^{\mathfrak{A}}(\bar{\varphi}(b_1), \dots, \bar{\varphi}(b_n))$. Let $C = \{b_1, \dots, b_n\}$. For the finite set C , there exists ψ in Σ with the domain C such that $\bar{\varphi}(b_i) = \psi(b_i)$, $i = 1, \dots, n$. Since ψ is a homomorphism, $R_i^{\mathfrak{B}}(b_1, \dots, b_n)$ implies $R_i^{\mathfrak{A}}(\psi(b_1), \dots, \psi(b_n))$, hence $R_i^{\mathfrak{A}}(\bar{\varphi}(b_1), \dots, \bar{\varphi}(b_n))$. Thus, $\bar{\varphi}$ is a homomorphism from \mathfrak{B} into \mathfrak{A} .

References

- [1] A. Robinson: On the construction of models. Essays on the Foundations of Mathematics (Fraenkel anniversary volume) (1961).
- [2] G. Grätzer: Universal Algebra. Van Nostrand (1968).