# 206. *Remark on Fixed Point of k-regular Mappings*

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The main purpose of this paper is to answer the question raised in [4]. The dilation $D_k$ of Euclidean $n$-space $R^n$ defined by $x \mapsto kx$ for some $k \in (0, 1)$ can be extended uniquely to the $n$-sphere, $S^n = R^n \cup \{\infty\}$. If $h$ is a homeomorphism of $S^n$ of the same topological type as $D_k$, then $h$ is regular except at two points. Kérekjartó [6], Homma and Kinoshita [2] showed the converse for $n=2$, $n=3$ respectively. Husch [3] extended Homma and Kinoshita's result for $n \geqq 6$. He [4] considered the topological characterization of the dilation in a separable infinite dimensional Fréchet space $E$ (i.e. in a separable infinite dimensional locally convex complete linear metric space).

In [4], Husch has the following theorems. Let $h$ be a homeomorphism of $E$ (with metric $d$) onto itself.

**Theorem** (Husch [4]). *Suppose that $h$ is $k$-regular at each point of $E$, $0<k<1$ (i.e. for each $\varepsilon>0$, there exists $\delta>0$ such that if $d(x, y)<\delta$, then $d(h^n(x), h^n(y))<k^n\varepsilon$ for each integer $n$).*

(1) ([4], Proposition 6, p. 4) *$h$ has at most one fixed point.*

(2) ([4], Theorem 1, p. 2) *If the fixed point set of $h$, Fix $(h)$, is not empty, then $h$ has the topological type of a dilation $D_k$.*

(3) ([4], Theorem 2, p. 2) *If Fix $(h)$ is empty, then $h$ has the topological type of a translation.*

In this paper we prove the following:

**Theorem 1.** *If $h$ is $k$-regular at each point of $E$, $0<k<1$, then $h$ has a unique fixed point.*

Hence we can eliminate the hypothesis that *Fix*$(h)$ be a non empty set in Husch's result (2).

Every separable infinite dimensional Fréchet space $E$ is homeomorphic to the countable infinite product of lines [1]. Hence $E$ is connected metric space. Thus we only show the following:

**Lemma 2.** *Let $h$ be a $k$-regular mapping, $(0<k<1)$, of a complete, connected metric space $X$ onto itself. Then $h$ has a unique fixed point.*

Before starting the proof, we recall the following definitions and some properties [5]. Let $h$ be a continuous mapping in a metric space $X$. If for each $\varepsilon>0$, there exists $n \in I^+$ (positive integers) such that

$$d(h^m(x), h^m(y))<\varepsilon \qquad \text{for all } m \geqq n,$$

then $x$ and $y$ are said to be *asymptotic under h*. (Abbreviate $x \sim y$). Then $\sim$ is an equivalence relation on $X$. Let $X_h$ be the set of all equivalence classes. $\bar{x}$ denotes the equivalence class of $x \in X$. The induced mapping $\bar{h}: X_h \to X_h$ is well defined as follows. For each $\bar{x} \in X_h$, $\bar{h}(\bar{x}) = \overline{h(x)}$. Then we have the following theorems.

**Theorem 3** (Kashiwagi and Maki [5], Theorem 12, p. 7). *Let X be a complete metric space. Then the continuous mapping h has a unique fixed point if and only if the induced mapping $\bar{h}$ has a unique fixed point.*

**Theorem 4** ([5], Theorem 13, p. 7). *Let all assumptions of Theorem 3 hold. If $X_h$ is a singleton, then h has a unique fixed point.*

**Proof of Lemma 2.** Since $h$ is $k$-regular at each point $x$, there exists a $\delta(x)$-neighbourhood $B_x(\delta(x))$, with center $x$ and radius $\delta(x)$ such that

$$\text{if} \quad \forall y \in B_x(\delta(x)), \qquad \text{then } x \sim y.$$

Let $x$ be any point of $X$. By the above discussion, $\bar{x}$ is open in $X$. And $\bar{x}$ is not empty. Note that $\bar{x}$ is a closed set in $X$. For suppose $\{x_n\}$ is a sequence in $\bar{x}$ such that

$$x_n \to a \quad \text{as} \quad n \to +\infty.$$

Since $h$ is $k$-regular at $a$, then there exists an integer $N$ such that

$$x_n \sim a \quad \text{for all } n > N. \qquad x, x_n \in \bar{x} \text{ implies } x_n \sim x \quad \text{for } n.$$

Hence we have $x \sim a$, $a \in \bar{x}$. This implies $\bar{x}$ is closed in $X$. Since $X$ is connected, $X = \bar{x}$. Hence $X_h$ is a singleton. With the use of Theorem 4, the proof is complete. Q.E.D.

**Remark 1.** Theorem 1 is the answer to the question raised in [4]. The hypothesis that *Fix* $(h)$ be empty can never be satisfied in Husch's result (3). Hence that lines should be delated from the theorem (Theorem 2 [4]).

Now, suppose that there exists an everywhere dense subset $Y$ of $X$. We have the following:

**Theorem 5.** *Let f be a k-regular mapping of a complete metric space X onto itself. If $Y_f$ is a singleton, then f has a unique fixed point.*

*Proof.* We show that $X_f$ is a singleton. Let $x$ be any point of $Y$, $y$ any point of $X-Y$. Then there exists a sequence $\{x_n\}$ of $Y$ such that

$$\lim_{n\to\infty} x_n = y.$$

Since $f$ is $k$-regular, $x_n \sim y$ for some integer $n$. Clearly $x_n \sim x$. Hence $x \sim y$ if $x \in Y$, $y \in X-Y$. Now let $x$, $y$ be any points of $X-Y$. Then there exist the sequences $\{x_n\}$, $\{y_n\}$ such that

$$\lim x_n = x \quad \text{and} \quad \lim y_n = y.$$

Since $x_n, y_n \in Y$, $y, x \in X-Y$, then $x_n \sim x$ and $y_m \sim y$ for some integers $n, m$. Since $Y_f$ is a singleton, we have $x \sim y$. This implies $X_f$ is a singleton. Q.E.D.

**Remark 2.** If one replaces the condition that $f$ is $k$-regular with the condition that $f$ is continuous, then the resulting proposition need not be true, as the following example shows. Define $f$ on the interval $X=[-\sqrt{2}, +\infty)$ as follows:

$f(x)=2x+\sqrt{2}$ *if* $-\sqrt{2}\leqq x\leqq\sqrt{2}$, $f(x)=x/2+5/\sqrt{2}$ *if* $x>\sqrt{2}$.

$f$ is not $k$-regular, $0<k<1$, and $Y_f$ is a singleton where $Y=Q\cap X$. But $f$ has two fixed points.

In the end of this paper, we give another application of Theorem 4, which treat a subject of Kannan's fixed point theorem in metric space.

**Theorem 6.** *Let $X$ be a complete metric space. Let $f$ be a continuous mapping of $X$ into itself such that*

$d(f(x), f(y))\leqq \alpha d(x, f(x))+\beta d(y, f(y))+\gamma d(x, y)$ *where* $x, y\in X$ *and* $0<\alpha+\beta+\gamma<1$, $0\leqq\alpha$, $0\leqq\beta<1$, $0\leqq\gamma<1$. *Then $f$ has a unique fixed point.*

**Proof.** Let $x$, $y$ be any point of $X$. In order to complete the proof, we see $x\sim y$. For all $n$ we have

$$d(f^n(x), f^{n+1}(x))\leqq\left(\frac{\alpha+\gamma}{1-\beta}\right)^n d(x, f(x)).$$

$$d(f^n(x), f^n(y))\leqq \alpha d(f^{n-1}(x), f^n(x))+\beta d(f^{n-1}(y), f^n(y)) +\gamma d(f^{n-1}(x), f^{n-1}(y)).$$

Hence we have

$$d(f^{n+1}(x), f^{n+1}(y))\leqq\alpha\left(\frac{\alpha+\gamma}{1-\beta}\right)^n d(x, f(x))+\beta\left(\frac{\alpha+\gamma}{1-\beta}\right)^n d(y, f(y)) +\gamma d(f^n(x), f^n(y)).$$

By the induction,

$$d(f^{n+1}(x), f^{n+1}(y))\leqq\left(\sum_{i=0}^{n}\gamma^i\left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-i}\right)\{\alpha d(x, f(x))+\beta d(y, f(y))\} +\gamma^{n+1}d(x, y).$$

Let $B_n=\sum_{i=0}^{n}\gamma^i\left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-i}$. We have $\lim_{n\to\infty} B_n=0$. Therefore

$$d(f^n(x), f^n(y))\to 0 \qquad \text{as } n\to+\infty.$$

This implies that $X_f$ is a singleton. By Theorem 4 the proof is complete. Q.E.D.

**Remark 3.** By Theorem 6, we have the Banach's fixed point Theorem and Kannan's result [7].

**Added in proof.** Some changes need in Theorem 3 and Theorem 4. $\bar{h}$ has a unique fixed point $\bar{x}$ with a Cauchy sequence $\{h^n(x)\}$, if and only if, $h$ has a unique fixed point. However, we assume that there exists a Cauchy sequence $\{h^{n_i}(x)\}$ for some $x$. In this case, if $X_h$ is a singleton, then $h$ has a unique fixed point. Therefore, in this case

Theorem 1, Lemma 2 and Theorem 5 are valid. Thus Remark 1 and two lines (p. 925, lines 24, 25) in this paper, should be deleted.

## References

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