205. *On a Non-linear Volterra Integral Equation with Singular Kernel*

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In the present paper we consider the solution $y(x)$ of the non-linear Volterra integral equation

$$(1) \qquad y(x)=f(x)+\int_0^x p(x,t)k(x,t,y(t))dt$$

where $p(x,t)$ is supposed to be unbounded in the region of integration.

Examples. $p(x,t)=(x-t)^{-1/2}$, or $p(x,t)=t(x^2-t^2)^{-1/2}$.

Evans [1] studied a similar problem using the convolution. Our treatment below is more elementary than his. We also consider the continuity and differentiability with respect to a parameter of solutions of (1) when it contains a parameter.

1. Existence theorem. In equation (1) we shall assume the four conditions:

(a) $f(x)$ is continuous in the interval I_a,

$I_a=\{x \mid 0\leqslant x\leqslant a\}$;

(b) $k(x,t,y)$ is continuous in the region Δ,

where $\Delta=\{(x,t,y) \mid 0\leqslant t\leqslant x\leqslant a,\ |y-f(x)|\leqslant b\}$,

$$\sup_{0\leqslant t\leqslant x\leqslant a} k(x,t,f(x))=K,$$

$k(x,t,y)$ satisfies a Lipschitz condition:

$$|k(x,t,y_1)-k(x,t,y_2)|\leqslant L|y_1-y_2|;$$

(c) $\int_0^x |p(x,t)|\,dt\leqslant M<\infty \qquad (0\leqslant x\leqslant a)$;

(d) for any $\varepsilon>0$, there exists $\delta>0$, independent of x and α, such that

$$\int_\alpha^{\alpha+\delta} |p(x,t)|\,dt<\varepsilon \qquad \text{for all } 0\leqslant\alpha\leqslant x-\delta.$$

Theorem 1. *Under the conditions* (a), (b), (c), (d), *equation* (1) *has a unique continuous solution on the interval* $0\leqslant x\leqslant h$, *where* h *is determind as follows*:

for any ρ, $0<\rho<1$, *let* $P=\min\left(\frac{\rho}{L},\frac{b}{K}\right)$ *and then let* $h=\min(r,a)$,

where r *is determined by*

$$\int_0^x |p(x,t)|\,dt\leqslant P \qquad (0\leqslant x\leqslant r).$$

Proof. For $n=1,2,\cdots$, let us put

$$y_n(x)=f(x)+\int_0^x p(x,t)k(x,t,y_{n-1}(t))dt,$$

where $y_0(x)=f(x)$.

Then, by our determination of P and h, $y_n(x)$ is defined in I_h and satisfies the inequality

$$|y_n(x)-f(x)|\leqslant b \qquad (n=1,2,\cdots).$$

By Lipschitz condition, for $x\in I_h$,

$$\begin{aligned}|y_{n+1}(x)-y_n(x)|&\leqslant L\int_0^x |p(x,t)|\,|y_n(t)-y_{n-1}(t)|\,dt\\ &\leqslant L\sup_{x\in I_h}|y_n(x)-y_{n-1}(x)|\int_0^x|p(x,t)|\,dt\\ &\leqslant LP\sup_{x\in I_h}|y_n(x)-y_{n-1}(x)|.\end{aligned}$$

Hence we have

$$|y_{n+1}(x)-y_n(x)|\leqslant b(LP)^n\leqslant b\rho^n \qquad (n=0,1,2,\cdots).$$

Since $0<\rho<1$, the sequence $\{y_n(x)\}$ is uniformly convergent in I_h and $y(x)=\min_{n\to\infty} y_n(x)$ is a continuous solution of equation (1).

The uniqueness follows from the Lipschitz condition on $k(x,t,y)$.

2. Prolongation of solution.

Theorem 2. *The solution curve of equation* (1) *in Theorem* 1 *can be prolonged to the terminal point $x=a$ of the interval I_a, when $k(x,t,y)$ is defined and continuous in $0\leqslant t\leqslant x\leqslant a$, $|y|<\infty$.*

Proof. Let $y=\bar{y}(x)$ be the solution of (1) in the interval I_h, $0\leqslant x\leqslant h(<a)$, where $h=\min(r,a)$ and r is determined by

$$\int_0^x|p(x,t)|\,dt\leqslant P=\frac{\rho}{L} \qquad (0\leqslant x\leqslant r).$$

Then, from Theorem 1 we know that the equation

$$y(x)=f(x)+\int_0^h p(x,t)k(x,t,\bar{y}(t))dt+\int_h^x p(x,t)k(x,t,y(t))dt$$

has a continuous solution $y=\bar{\bar{y}}(x)$ in the interval $h\leqslant x\leqslant 2h$ $(\leqslant a)$, because $\int_0^h p(x,t)k(x,t,\bar{y}(t))dt$ is bounded and continuous.

The function

$$y(x)=\begin{cases}\bar{y}(x) & 0\leqslant x\leqslant h\\ \bar{\bar{y}}(x) & h\leqslant x\leqslant 2h\end{cases}$$

is the solution of (1) in the interval I_{2h}.

Repeating the same procedure finite times, we can reach to $x=a$.

3. Equation containing a parameter. Let us consider the integral equation

$$(2) \qquad y(x)=f(x,\lambda)+\int_0^x p(x,t)k(x,t,y(t),\lambda)dt$$

containing a parameter λ.

Theorem 3. *In equation* (2), *we shall assume the following* *conditions*:

(a′) *$f(x, \lambda)$ is continuous in the region*
$$\{(x, \lambda) \mid 0 \leqslant x \leqslant a, |\lambda| \leqslant l\};$$

(b′) *$k(x, t, y, \lambda)$ is bounded continuous in the region*
$$\{(x, t, y, \lambda) \mid 0 \leqslant t \leqslant x \leqslant a, |y - f(x, \lambda)| \leqslant b, |\lambda| \leqslant l\}$$
and satisfies a Lipschitz condition
$$|k(x, t, y_1, \lambda) - k(x, t, y_2, \lambda)| \leqslant L\,|y_1 - y_2|;$$

(c) $\int_0^x |p(x, t)|\, dt \leqslant M < \infty \qquad (0 \leqslant x \leqslant a);$

(d′) *for any $\varepsilon > 0$, there exists a $\delta > 0$, independent of α and x, such that*
$$\int_\alpha^{\alpha+\delta} |p(x, t)|\, dt < \varepsilon \qquad \textit{for all } 0 \leqslant \alpha \leqslant x - \delta.$$

Further suppose that for any λ ($|\lambda| \leqslant l$) there exists a (unique) solution of (2) in the interval $I_a = \{x \mid 0 \leqslant x \leqslant a\}$.

Then the solution $y = y(x, \lambda)$ is continuous with respect to the parameter λ in the region $\{(x, \lambda) \mid 0 \leqslant x \leqslant a, |\lambda| \leqslant l\}$.

Proof. For any ρ, $0 < \rho < 1$, let $h = \min(r, a)$, where r is determined by $\int_0^x |p(x, t)|\, dt \leqslant \frac{\rho}{L}$ $(0 \leqslant x \leqslant r)$.

First we consider the solution $y(x, \lambda)$ in the region $\{(x, \lambda) \mid 0 \leqslant x \leqslant h, |\lambda| \leqslant l\}$.

For any $\lambda + \Delta\lambda$, $|\lambda + \Delta\lambda| < 1$, we have
$$\begin{aligned} &y(x, \lambda + \Delta\lambda) - y(x, \lambda) \\ &\quad = f(x, \lambda + \Delta\lambda) - f(x, \lambda) \\ &\qquad + \int_0^x p(x, t)\{k(x, t, y(t, \lambda + \Delta\lambda), \lambda + \Delta\lambda) - k(x, t, y(t, \lambda), \lambda + \Delta\lambda)\} dt \\ &\qquad + \int_0^x p(x, t)\{k(x, t, y(t, \lambda), \lambda + \Delta\lambda) - k(x, t, y(t, \lambda), \lambda)\} dt, \end{aligned}$$
and therefore
$$\begin{aligned} &|y(x, \lambda + \Delta\lambda) - y(x, \lambda)| \\ &\quad \leqslant \delta_1(\Delta\lambda) + L\int_0^x |p(x, t)|\,|y(t, \lambda + \Delta\lambda) - y(t, \lambda)|\, dt + M\delta_2(\Delta\lambda), \end{aligned} \tag{3}$$
where $|f(x, \lambda + \Delta\lambda) - f(x, \lambda)| \leq \delta_1(\Delta\lambda)$,
$$|k(x, t, y, \lambda + \Delta\lambda) - k(x, t, y, \lambda)| \leqslant \delta_2(\Delta\lambda).$$
Let
$$\delta_1(\Delta\lambda) + M\delta_2(\Delta\lambda) = \delta(\Delta\lambda),$$
then, from assumptions (a′) and (b′)
$$\delta(\Delta\lambda) \to 0 \quad \text{as} \quad \Delta\lambda \to 0.$$
Let
$$\sup_{(x, \lambda) \in I_h \times \Lambda} |y(x, \lambda + \Delta\lambda) - y(x, \lambda)| = \delta(y),$$
then from (3)
$$\delta\,(y) \leqslant \delta(\Delta\lambda) + \rho\delta(y)$$

or

$$\delta(y) \leqslant \delta(\Delta\lambda)/(1-\rho).$$

Therefore, when $\Delta\lambda \to 0$, $\delta(\Delta\lambda) \to 0$ and $\delta(y) \to 0$, that is, $y(x, \lambda)$ is continuous with respect to λ.

Applying the same argument successively, we can prove the continuity of $y(x, \lambda)$ with respect to λ on the whole interval I_a. Q.E.D.

From Theorem 3, we have evidently

Theorem 4. *Suppose the equation* (2) *satisfies the following conditions*:

(a″) $f(x, \lambda)$, $\dfrac{\partial}{\partial \lambda} f(x, \lambda)$ *are continuous in the region*

$$\{(x, \lambda) \mid 0 \leqslant x \leqslant a,\ |\lambda - \lambda_0| \leqslant l\}.$$

(b″) $k(x, t, y, \lambda)$, $\dfrac{\partial}{\partial y} k(x, t, y, \lambda)$, $\dfrac{\partial}{\partial \lambda} k(x, t, y, \lambda)$

are continuous in the region

$$\{(x, t, y, \lambda) \mid 0 \leqslant x \leqslant t \leqslant a,\ |y - f(x, \lambda)| \leqslant b,\ |\lambda - \lambda_0| \leqslant l\};$$

(c) $\displaystyle\int_0^x |p(x, t)|\, dt \leqslant M < \infty \qquad (0 \leqslant x \leqslant a)$

(d) *for any* $\varepsilon > 0$, *there exists a* $\delta > 0$, *independent of* α, x, λ, *such that*

$$\int_\alpha^{\alpha+\delta} |p(x, t)|\, dt < \varepsilon \qquad \text{for all } 0 \leqslant \alpha \leqslant x - \delta.$$

Further suppose that for any λ, $|\lambda - \lambda_0| \leqslant l$, *there exists a* (*unique*) *solution* $y(x, \lambda)$ *of* (2) *in the interval* I_a.

Then the solution $y(x, \lambda)$ *is continuously differentiable with respect to* λ *at* $\lambda = \lambda_0$.

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Reference

[1] G. C. Evans: Volterra's integral equation of the second kind with discontinuous kernel. Trans. Amer. Math. Soc., **11**, 393–413 (1910).