130. On Pseudoparacompactness and Continuous Mappings

By Takao HOSHINA
Tokyo University of Education

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Throughout this paper we assume that spaces are completely regular T_1-spaces and maps are continuous. The completion of a space X with respect to its finest uniformity is called the topological completion of X, and denoted by μX. According to Morita [8] a space X is called pseudoparacompact (resp. pseudo-Lindelöf) if μX is paracompact (resp. Lindelöf).

As for these notions, in the same paper Morita proved the following remarkable results.

Theorem 1 (Morita [8], Theorems 3.1, 3.2 and 3.5).

(1) *μX is compact iff X is pseudocompact.*

(2) *μX is always a paracompact M-space for any M-space X.*

(3) *Let X be an M-space. X is pseudo-Lindelöf iff it is the quasi-perfect inverse image of a separable metric space.*

The characterizations of pseudoparacompactness and pseudo-Lindelöfness have been obtained by Howes [4] and Ishii [5] independently. On the other hand, in [2] Hanai and Okuyama (cf. Isiwata [6]) essentially proved the following result: "*If a space X is the inverse image of a pseudocompact space under an open quasi-perfect map, then X is pseudocompact*". Here the assumption that the map is open cannot be dropped in general ([3] Example 2.4). Analogously to this result, in § 1 we shall prove the following theorem which is a partial answer to a problem posed by Ishii [5] concerning (2) and (3) of Theorem 1: "*Is pseudoparacompactness or pseudo-Lindelöfness preserved under taking the inverse image by a quasi-perfect* (*or perfect*) *map?*"

Theorem 2. *If there is an open quasi-perfect map $\varphi: X \to Y$ from a space X onto a pseudoparacompact* (*resp. pseudo-Lindelöf*) *space Y, then X is pseudoparacompact* (*resp. pseudo-Lindelöf*).

In § 2, by virtue of recent results obtained by Morita, we shall prove the following

Theorem 3. *Let $\varphi: X \to Y$ be an open quasi-perfect map from a space X onto a space Y.*

(1) *If μY is locally compact and paracompact, then so is μX.*

(2) *If μY is σ-compact, then so is μX.*

§ 1. Proof of Theorem 2. Before proving Theorem 2, we shall need some preliminalies. For a space X, let μ be the finest uniformity of X and ν the uniformity of all countable normal coverings of X.

Lemma 1.1 (Howes [4]). *A space X is pseudoparacompact (resp. pseudo-Lindelöf) iff for any weakly Cauchy filter $\mathfrak{F}$ with respect to μ (resp. ν) there exists a Cauchy filter $\mathfrak{G}$ with respect to μ containing $\mathfrak{F}$.*

Here a filter $\mathfrak{F}$ in X is called weakly Cauchy with respect to a uniformity μ of X if for any uniform cover $\mathfrak{U}$ in μ there is a filter $\mathfrak{G}$ in X containing $\mathfrak{F}$ such that $G\subset U$ holds for some $G\in\mathfrak{G}$ and $U\in\mathfrak{U}$.

Let $\mathcal{C}(X)$ be the family of all non-empty compact subsets of a given space X. Following the convention of [7], we topologize $\mathcal{C}(X)$ with the Vietoris topology; for a finite collection $\{U_1, U_2, \cdots, U_n\}$ of open sets, $\langle U_1, U_2, \cdots, U_n\rangle$ will denote the subset of $\mathcal{C}(X)$ to which the compact set K belongs iff $K\subset\bigcup_i U_i$ and $K\cap U_i\neq\emptyset$ for $i=1,2,\cdots,n$. Open sets in $\mathcal{C}(X)$ are unions of an arbitrary number of these sets.

Lemma 1.2 (Michael [7]). *$\mathcal{C}(X)$ is completely regular and T_1 iff X is completely regular and T_1.*

A space X is called topologically complete if $\mu X=X$ (cf. [8]).

Lemma 1.3 (Zenor [10]). *$\mathcal{C}(X)$ is topologically complete iff X is topologically complete.*

A subset F of a space X is called relatively pseudocompact if every real-valued continuous function over X is bounded on F.

Lemma 1.4 (Dykes [1]). *If F is a relatively pseudocompact subset of a topologically complete space X, then $\mathrm{cl}_X F$ is compact.*

A map $\varphi: X\to Y$ is called a Z-map if the image of each zero-set in X is closed in Y. In [6], Isiwata extended the notion of Z-maps; a map $\varphi: X\to Y$ is a WZ-map if $\mathrm{cl}_{\beta X}\varphi^{-1}(y)=\beta(\varphi)^{-1}(y)$ for every y in Y, where $\beta(\varphi)$ denotes the Stone extension of φ.

The following lemma is useful.

Lemma 1.5. *Let $\varphi: X\to Y$ be a map from X onto Y such that $\varphi^{-1}(y)$ is relatively pseudocompact for each y in Y. For y in Y, let us put $\tilde{\varphi}(y)=\mathrm{cl}_{\mu X}\varphi^{-1}(y)$. If φ is an open WZ-map, then the mapping $\tilde{\varphi}$ from Y into $\mathcal{C}(\mu X)$ is continuous. Conversely if $\tilde{\varphi}$ is continuous then φ is open, and moreover if X is normal then φ is closed.*

Proof. Clearly $\tilde{\varphi}$ maps Y into $\mathcal{C}(\mu X)$ by Lemma 1.4. Let φ be an open WZ-map, and for y in Y let $\tilde{\varphi}(y)\in\langle U_1, U_2, \cdots, U_n\rangle$, where U_i is an open set in μX for each i. If we choose an open set U' in βX such that $U'\cap\mu X=\bigcup_i U_i$, then the set $V=\bigcap_i\varphi(U_i\cap X)\cap(\beta Y-\beta\varphi(\beta X-U'))$ is an open set in Y containing y since φ is an open WZ-map and $\mathrm{cl}_{\mu X}\varphi^{-1}(y)$ is compact. Moreover we easily see $\tilde{\varphi}(V)\subset\langle U_1, U_2, \cdots, U_n\rangle$. Therefore $\tilde{\varphi}$ is continuous. Conversely let us assume $\tilde{\varphi}$ is continuous. Let U be an open set in X and choose an open set U' in μX such that $U'\cap X=U$. Then $\varphi(U)=\tilde{\varphi}^{-1}(\langle U', \mu X\rangle\cap\tilde{\varphi}(Y))$. Hence φ is open. Now, let us assume X is normal. For a closed set F in X, let us put $\mathfrak{F}=\{K \in \mathcal{C}(\mu X) \mid \mathrm{cl}_{\mu X}F \cap K \neq \emptyset\}$. Then $\mathfrak{F}$ is closed in $\mathcal{C}(\mu X)$, and $\varphi(F) = \tilde{\varphi}^{-1}(\mathfrak{F} \cap \tilde{\varphi}(Y))$. Therefore φ is closed. This proves Lemma 1.5.

Theorem 2 is an immediate consequence of the following

Theorem 4. *Let $\varphi: X \to Y$ be an open WZ-map from a space X onto a pseudoparacompact (resp. pseudo-Lindelöf) space Y such that $\varphi^{-1}(y)$ is relatively pseudocompact for each y in Y, then X is pseudoparacompact (resp. pseudo-Lindelöf).*

Proof. Let $\mathfrak{F}$ be a weakly Cauchy filter in X with respect to μ (resp. ν). Then the filter $\varphi(\mathfrak{F})$ is weakly Cauchy with respect to μ (resp. ν) since φ is continuous. Moreover since Y is pseudoparacompact (resp. pseudo-Lindelöf), by Lemma 1.1 there exists a Cauchy filter $\mathfrak{G}$ in Y with respect to μ, which contains $\varphi(\mathfrak{F})$. Let $\tilde{\varphi}$ be a map as in Lemma 1.5, then $\tilde{\varphi}(\mathfrak{G})$ is also a Cauchy filter in $\mathcal{C}(\mu X)$ with respect to μ since $\tilde{\varphi}$ is continuous. Therefore since $\mathcal{C}(\mu X)$ is topologically complete by Lemma 1.3, $\tilde{\varphi}(\mathfrak{G})$ converges to some K in $\mathcal{C}(\mu X)$. Let us suppose that $\bigcap\{\mathrm{cl}_{\mu X}F \mid F \in \mathfrak{F}\} \cap K = \emptyset$. Since K is compact, it follows that $\mathrm{cl}_{\mu X}F \cap K = \emptyset$ for some $F \in \mathfrak{F}$. This means that $K \in \langle \mu X - \mathrm{cl}_{\mu X}F \rangle$. Since $\tilde{\varphi}(\mathfrak{G})$ converges to K, there exists G in $\mathfrak{G}$ such that $\tilde{\varphi}(G) \subset \langle \mu X - \mathrm{cl}_{\mu X}F \rangle$. Then it is easily seen that $\varphi^{-1}G \subset X - F$. But this contradicts that $\varphi(\mathfrak{F}) \subset \mathfrak{G}$. Hence $\mathfrak{F}$ has a cluster point in K. This shows that $\mathfrak{F}$ is contained in a Cauchy filter in X with respect to μ. Therefore X is pseudoparacompact (resp. pseudo-Lindelöf) by Lemma 1.1. The proof is completed.

Remark. Under the map $\varphi: X \to Y$ given in Theorem 4, let us assume that Y is pseudocompact and consider $\mathfrak{F}$ in the proof above to be a weakly Cauchy filter with respect to the uniformity of all finite normal coverings, then under the same argument as above, by ([4], Theorem 3) we can conclude that X is pseudocompact. This is an another proof of ([6], Theorem 4.2).

As an application of Theorem 4 we have

Theorem 5. *Let X be a pseudocompact space and Y a first countable and pseudoparacompact (resp. pseudo-Lindelöf) space. Then $X \times Y$ is pseudoparacompact (resp. pseudo-Lindelöf).*

Proof. Since the projection $X \times Y \to Y$ is a Z-map by ([6], Theorem 2.1), this follows from Theorem 4.

§2. Proof of Theorem 3. Theorem 3 is a direct consequence of the following lemma and theorems which are due to Morita.

Lemma 2.1. *Let $\varphi: X \to Y$ be an open WZ-map from X onto Y such that $\varphi^{-1}(y)$ is relatively pseudocompact for each y in Y. If F is a relatively pseudocompact subset of Y, then $\varphi^{-1}(F)$ is relatively pseudocompact.*

Proof. For any real-valued continuous function f on X, let us define real-valued functions f^s and f^i on Y by

$$f^s(y)=\sup\{f(x)\mid x\in\varphi^{-1}(y)\},\qquad f^i(y)=\inf\{f(x)\mid x\in\varphi^{-1}(y)\}.$$

Then f^s and f^i are continuous by ([6], Lemma 4.1) and bounded on F. Hence f is bounded on $\varphi^{-1}(F)$ and this proves Lemma 2.1.

Theorem 6 (Morita [9]). *For a space X, μX is locally compact and paracompact iff there exists a normal open covering of X consisting of relatively pseudocompact subsets.*

Theorem 7 (Morita). *For a space X, μX is σ-compact iff X is expressed as a union of a countable number of relatively pseudocompact subsets.*

Proof. Let $\mu X=\cup\{K_i\mid i=1,2,\cdots\}$, where each K_i is compact. Then $X=\bigcup_i (K_i\cap X)$ and since X is C-embedded in X by ([8], Theorem 2.4), $K_i\cap X$ is relatively pseudocompact. Conversely, suppose that $X=\cup\{F_i\mid i=1,2,\cdots\}$, where each F_i is relatively pseudocompact. Let us put $Y=\bigcup_i \mathrm{cl}_{\mu X}F_i$. Then $X\subset Y\subset\mu X$ and Y is a σ-compact space by Lemma 1.4. Therefore by ([8], Theorem 2.5) it holds that $Y=\mu X$. Hence μX is σ-compact and this completes the proof of Theorem 7.

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