129. *One Condition for* R(K)=A(K)

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We will show here one condition to coincide $A(K)$, all continuous functions on the compact plane set K which are analytic in $\mathring{K}$, and $R(K)$, all those functions on K which are approximable by rational functions with poles off K. This sharpens the result of Theorem 4.1 in [3].

Let U be a bounded open set in the complex plane C, $\overline{U}$ be the closure of U, and ∂U be the boundary of U. Let $A(U)$ be all continuous functions on $\overline{U}$ which are analytic in U and $R(U)$ be all those functions which are approximable uniformly on $\overline{U}$ by rational functions with poles off $\overline{U}$. Let $H^\infty(U)$ be the uniform algebra of all bounded analytic functions on U.

Lemma 1. *Let B be a subalgebra in $H^\infty(U)$ which contains $A(U)$. Then there is a continuous map from the maximal ideal space M_B of B onto $\overline{U}$.*

Proof. The coordinate function Z belongs to B and the Gelfand transform $\hat{Z}$ of Z is the desired map. For since $B \subseteq H^\infty(U)$, every homomorphism in the maximal ideal space of $H^\infty(U)$ determines a homomorphism in M_B by restricting it to B. So $\hat{Z}(M_B)$ contains $\overline{U}$. Suppose $\lambda \notin \overline{U}$, then $(z-\lambda)^{-1} \in A(U)$, that is, $z-\lambda$ is invertible in B. Thus $\varphi(z-\lambda) \neq 0$ for all $\varphi \in M_B$. Hence λ does not belong to $\hat{Z}(M_B)$ and $\hat{Z}(M_B)=\overline{U}$. This completes the lemma.

The analogous result is valid by replacing the algebra $A(U)$ by the algebra $R(U)$.

For B as above, we denote the fibers $M_\lambda(B)$ of M_B over points $\lambda \in \overline{U}$ by

$$M_\lambda(B)=\{\varphi \in M_B\,;\, \varphi(z)=\lambda\}.$$

If $\lambda \in U$, then $M_\lambda(B)$ consists of the single homomorphism.

Lemma 2. *Let B be as above lemma. Then for each $\lambda \in \partial U$ and for each $f \in A(U)$, $\varphi(f)=f(\lambda)$ for all $\varphi \in M_\lambda(B)$.*

Proof. As seen in [2], by using the Vitushkin's operator, we can find a bounded sequence $f_n \in A(U)$ which is analytic at $\{\lambda\}$ and the f_n converges uniformly to f on $\overline{U}$. So it is sufficient to show the case that $g \in A(U)$ is analytic at $\{\lambda\}$. If $g \in A(U)$ is analytic at $\{\lambda\}$, then

$$\frac{g(z)-g(\lambda)}{z-\lambda} \in A(U). \quad \text{Hence } \frac{g(z)-g(\lambda)}{z-\lambda} \in B. \quad \text{Thus } \varphi(g)=g(\lambda)$$

for all $\varphi \in M_B$ and $\varphi(z)=\lambda$. And the lemma is proved.

We define a subalgebra B of $H^\infty(U)$ by

$$B=\{f\in H^\infty(U)\,;\ \text{some } f_n\in R(U) \text{ bounded sequence } f_n(z)\to f(z) \text{ pointwise for all } z\in U.\} \qquad (*)$$

Let μ_z denote harmonic measure for $z\in U$ on ∂U. We denote the positive measure μ on ∂U by

$$\mu=\sum_i \frac{1}{2^i}\mu_i$$

where μ_i harmonic measure for some fix point $z_i\in U_i$, the components of U. Let σ be a measure on ∂U and let $H^\infty(\mu+|\sigma|)$ be the weak-star closure of $R(U)$ in $L^\infty(\mu+|\sigma|)$.

Lemma 3. *Suppose B is $(*)$ and $C(\partial U)=R(\partial U)$.*

Then the operator $f\to\tilde f(z)=\int f\,d\mu_z$, $z\in U$, is an isomorphism between a subalgebra H_σ in $H^\infty(\mu+|\sigma|)$ for any annihilating measure σ on ∂U for $R(U)$, and B. Moreover, let M be the maximal ideal space of $L^\infty(\mu+|\sigma|)$ and $\check Z$ be the Gelfand transform of the coordinate function Z in $L^\infty(\mu+|\sigma|)$.

Then for $\lambda\in\partial U$ and for $f\in H_\sigma$, $f(\check Z^{-1}(\lambda))\subseteq f(M_\lambda(B))$.

Proof. We already know the fact that the condition $C(\partial U)=R(\partial U)$ implies that $f\to\tilde f(z)$, $z\in U$, is the continuous, one to one map from $H^\infty(\mu+|\sigma|)$ into $H^\infty(U)$. $\{\tilde f\,;\ f\in H^\infty(\mu+|\sigma|)\}$ contains B. For if $f\in B$, then there is a bounded sequence $f_n\in R(U)$ which converges pointwise to f on U. Then by its boundedness, f_n, regarded as an element in $H^\infty(\mu+|\sigma|)$, converges weak-star to some $g\in H^\infty(\mu+|\sigma|)$, and $\tilde g=f$ on U. So if we put $H_\sigma=\{f\in H^\infty(\mu+|\sigma|)\,;\ \tilde f\in B\}$. Then there is an isomorphism between H_σ and B. Hence we can define a map $S: M\to M_B$ by $\varphi(f)=f(S\varphi)$ when $\varphi\in M$ and $f\in H_\sigma$. Since Z maps K_B onto $\bar U$ by Lemma 1, $\check Z=Z\circ S$. Thus $f(Z^{-1}(\lambda))\subseteq f(Z^{-1}(\lambda))=f(M_\lambda(B))$, and the lemma is proved.

Theorem 4. *The following are equivalent.*

(1) *$R(U)=A(U)$.*

(2) *Each $f\in A(U)$ is pointwise boundedly approximable on U by $R(U)$. $C(\partial U)=R(\partial U)$.*

Proof. It is clear that (1) implies (2). So we will only show that (2) implies (1). Let B be $(*)$. Then $B\supseteq A(U)$. It suffices to show that $A(U)\subseteq H^\infty(|\sigma|)$, the weak-star closure of $R(U)$ in $L^\infty(|\sigma|)$, for any annihilating measure σ on ∂U for $R(U)$. Let σ be as above and $F\in A(U)$. By Lemma 3, the operator $f\to\tilde f(z)=\int f\,d\mu_z$, $z\in U$, is an isomorphism between an algebra H_σ in $H(\mu+|\sigma|)$ and B. Choose $f\in H_\sigma$ such that $\tilde f=F$ on U. Let φ belong to the maximal ideal space M of $L^\infty(\mu+|\sigma|)$. Then by Lemma 2, F, regarded as an element of $H^\infty(U)$, assumes the constant value $F(\check Z(\varphi))$ on the fiber $M_{\check Z(\varphi)}(B)$. So again by Lemma 3, we obtain $\varphi(f)=\varphi(F)$ for all $\varphi \in M$. Thus f and F coincide, regarded as functions in $L^\infty(\mu+|\sigma|)$ and $F \in H \subseteq H^\infty(\mu+|\sigma|)$. The theorem is proved.

Corollary 5. *Let K be a compact plane set. Then the following are equivalent.*

(1) $A(K)=R(K)$.

(2) a) *Each $f \in A(K)$ is pointwise boundedly approximable by $R(K)$ on $\mathring{K}$, the interior of K.*

b) $C(\partial U)=R(\partial U)$.

Proof. We define a subalgebra B in $H^\infty(\mathring{K})$ by

$$B=\{f \in H^\infty(\mathring{K})\,;\ \text{some } f_n \in R(K),\ \text{bounded sequence } f_n(z)\to f(z) \text{ pointwise for all } z \in \mathring{K}.\}$$

Then it will suffice to show that there is a continuous map from M_B, the maximal ideal space of B, onto $\bar{\mathring{K}}$, the closure of $\mathring{K}$, when K and $\bar{\mathring{K}}$ do not coincide [3]. The coordinate function Z belongs to B and by the method of the proof in Lemma 1, it is clear that $Z(M_B)\supseteq \bar{\mathring{K}}$. If $\lambda \notin \bar{\mathring{K}}$, then there is an open neighborhood V of λ such that V and $\bar{\mathring{K}}$ are disjoint. It follows that the component of $C-\bar{\mathring{K}}$ which contains λ intersects with a component of $C-K$. Hence $(z-\lambda)^{-1}$ is uniformly approximable by $R(K)$ on $\bar{\mathring{K}}$. So $(z-\lambda)^{-1}\subseteq B$. Thus $z-\lambda$ is invertible in B and λ does not belong to $\hat{Z}(M_B)$. It concludes $\hat{Z}(M_B)=\bar{\mathring{K}}$. The rest is proved by the same as to in the theorem.

References

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