126. *Nondegeneracy and Discrete Models*

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1. Let u_g be a faithful unitary representation of a group G on a Hilbert space H. Arveson [1] introduced the *nondegeneracy* of $\{u_g\,;\,g\in G\}$ if there is a vector $\xi\in H$ such that

(1) $$(u_{g_i}\xi\,|\,u_{g_j}\xi)=0 \qquad (i\neq j)$$

for any finite subset $\{g_1, g_2, \cdots, g_n\}$ of G. Arveson's original definition is given for abelian groups. However, it is clear that his definition is valid for noncommutative case. If A is a C^*-algebra generated by $\{u_g\,;\,g\in G\}$, Arveson proved, then the *spectrum* $\sigma(A)$ of all characters of A is homeomorphic to the compact character group X of G if G is a nondegenerated abelian group. Hence, A is isomorphic to the algebra $C(X)$ of all continuous functions defined on X.

If G is a discrete group, then

$$u_g\xi(h)=\xi(g^{-1}h)$$

defines the regular representation u_g of G on $l^2(G)$. Clearly, $\{u_g\,;\,g\in G\}$ is nondegenerate since $\xi=\delta_1$ satisfies (1) for any finite subset of G, where

(2) $$\delta_g(h)=\begin{cases}1 & \text{for } h=g\\ 0 & \text{for } h\neq g.\end{cases}$$

Hence Arveson's theorem implies that the C^*-algebra $R(G)$ generated by the regular representation of an abelian discrete group G is isomorphic to $C(X)$. Therefore, Arveson's theorem is rephrased into

Theorem 1. *If a C^*-algebra A is generated by a nondegenerate faithful unitary representation of an abelian group G, then A is isomorphic to the operator group algebra $R(G)$ of G.*

On the other hand, there is an another characterization of $R(G)$ given by one of the authors, cf. [3]. A pair (G, f) of a group G and a positive definite function f on G is a *model* for a C^*-algebra A if the following conditions are satisfied:

1°. There is a faithful unitary representation u_g of G,

2°. $\{u_g\,;\,g\in G\}$ generates A,

3°. f is a faithful state of A with $f(g)=f(u_g)$,

4°. $f(g)=1$ if and only if $g=1$.

A model is *discrete* if f satisfies

(3) $$f(g)=\begin{cases}1 & \text{for } g=1\\ 0 & \text{for } g\neq 1.\end{cases}$$

(In discrete models, f is trivial; hence G is called simply a discrete model for A). Obviously, a discrete group G is a discrete model for $R(G)$. Conversely, the following theorem is valid:

Theorem 2. *If an abelian group G is a discrete model for a C^*-algebra A, then A is isomorphic to the operator group algebra $R(G)$.*

In the present note, the equivalence of two characterizations will be discussed in § 2. By this equivalence, Arveson's main theorem in [1] will be given an alternative proof in § 3. Finally, in § 4, a proof of Theorem 2 will be sketched.

2. In this section, the commutativity of groups is not essential.

Theorem 3. *If G is a discrete model for a C^*-algebra A, then $\{u_g\,;\,g \in G\}$ is nondegenerate in the representation of A induced by the state f in 3°.*

In the Gelfand-Naimark-Segal representation of A induced by f, there is a unit vector φ such as $f(a)=(a\varphi\,|\,\varphi)$ for $a \in A$, so that (3) implies

$$(u_g\varphi\,|\,u_h\varphi)=(u_{h^{-1}g}\varphi\,|\,\varphi)=f(h^{-1}g)=0$$

for $h \neq g$. Hence $\{u_g\,;\,g \in G\}$ is nondegenerate.

The following converse of Theorem 3 is essentially contained in Arveson [1]:

Theorem 4. *If a group $\{u_g\,;\,g \in G\}$ of unitary operators is nondegenerate, then the C^*-algebra A generated by G has a discrete model G.*

It is clearly sufficient to show that A has a state f satisfying (3). For any $g \neq 1$, let K_g be the set of all states of A satisfying

$$f(u_g)=0. \tag{4}$$

Then K_g is a convex weakly* closed subset in the compact set of all states of A. For a finite subset $\{g_1, g_2, \cdots, g_n\}$ of G, there is a unit vector φ which satisfies (1) by the nondegeneracy of G; hence

$$f(a)=(a\varphi\,|\,\varphi) \tag{5}$$

is a state of A which satisfies (4) for $g=g_i$, so that

$$f \in K_{g_1} \cap K_{g_2} \cap \cdots \cap K_{g_n},$$

and $\{K_g\,;\,g \in G\}$ has the finite intersection property. Therefore, there is a state f with

$$f \in \bigcap_{g \neq 1} K_g,$$

and f satisfies (4) for all $g \neq 1$.

Theorems 3 and 4 allow us to state that the nondegeneracy and the notion of discrete models for C^*-algebras are essentially equivalent.

3. The following theorem is a variant of Theorem 2:

Theorem 5. *If an abelian group G is a discrete model for a C^*-algebra A, then the spectrum $\sigma(A)$ of A is homeomorphic to the character group X of G.*

If G is nondegenerate abelian group of unitary operators on a

Hilbert space H, and if G generates a C^*-algebra A, then G is a discrete model for A by Theorem 4. Hence, by Theorem 5, the spectrum $\sigma(A)$ of A is homeomorphic to X. This proves the following theorem of Arveson [1]:

Theorem 6. *If an abelian unitary group G is nondegenerate, then the spectrum of the C^*-algebra generated by G is homeomorphic to the character group of G.*

As pointed out by Arveson [1], Theorem 6 implies a theorem of A. Ionescu-Tulcea which states that the spectrum of an ergodic automorphism of a nonatomic probability space covers the whole circle, cf. also [2].

4. In this section, a proof of Theorem 2 is given which is analogous to that of [3; III].

If G is an abelian group which is a discrete model for a C^*-algebra A. Via the Gelfand-Naimark-Segal representation of A induced by the state f in 3°, A is assumed to act on a Hilbert space H with a separating and generating unit vector φ with (5). Therefore $\{u_g\varphi\,;\, g \in G\}$ forms a complete orthonormal set in H. Let

$$w(u_g\varphi)=\delta_g.$$

Then w maps a complete orthonormal set of H onto a complete orthonormal set $\{\delta_g\,;\, g \in G\}$ of $l^2(G)$. Hence w is a unitary transformation which maps H onto $l^2(G)$. Put

$$a^\phi=waw^*$$

for $a \in A$. Then ϕ is a spatial isomorphism of A which maps u_g into v_g:

$$u_g^\phi=v_g$$

for every $g \in G$, where v_g is the regular representation of G on $l^2(G)$, since

$$u_g^\phi\delta_h=wu_gw^*\delta_h=wu_gu_h\varphi=wu_{gh}\varphi=\delta_{gh}=v_g\delta_h.$$

Since the regular representation v_g generates $R(G)$, the spatial isomorphism ϕ maps A onto $R(G)$, which is required.

It is noteworthy that the commutativity of the group plays no role in the above proof.

References

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