125. *Dependent Elements of an Automorphism of a C*-algebra*

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1. Introduction. Let A be a unital C^*-algebra. For an ($*$-preserving) automorphism α of A, an element a of A is called a *dependent element* of α if

(1) $ax = x^\alpha a$ for any $x \in A$.

If α is an inner automorphism of A induced by a, then clearly a is a dependent element.

In [5], Nakamura and Takeda recognized the importance of the following implication:

($*$) *If a is a dependent element of α then $a=0$.*

They proved, among many others, in a finite factor A α satisfies ($*$) if α is outer, using a sophisticated argument. Recently, Kallman [3] called, when A is a von Neumann algebra, α *freely acting* if ($*$) is satisfied. His definition of free action agrees with the usual one due to von Neumann if A is an abelian von Neumann algebra. He proved, among others, every automorphism of a von Neumann algebra is directly decomposed into the freely acting and inner parts.

In the present note, we shall study some properties of dependent elements of automorphisms on C^*-algebras. We shall show, by elementary calculations, dependent elements are normal and invariant under the automorphism, in § 2. We shall discuss some applications in § 3, which include a completely elementary proof of a theorem of Nakamura, Takeda and Kallman. In § 4, we shall give a few remarks, one of which is a slight improvement of a proof of a theorem of Kallman.

2. Dependent elements. We shall prove some elementary lemmas some of which are already known. In this section, we shall assume that A is a C^*-algebra with the center Z.

Lemma 1 (Kallman). *If a is a dependent element of an automorphism α of A, then a^*a and aa^* belong to Z.*

Proof. The following proof is a slight improvement of Kallman's. From (1), we have

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(2) $$xa^* = a^* x^\alpha$$

for all $x \in A$. Multiplying a by left in the both sides of (2), we have, for any $x \in A$,

$$aa^* x^\alpha = axa^* = x^\alpha aa^*,$$

so that we have $aa^* \in Z$. Similarly, by (1) and (2), we have

$$xa^* a = a^* x^\alpha a = a^* ax,$$

for every $x \in A$, so that we have $a^* a \in Z$.

Lemma 2. *If a is a dependent element of α, then a is normal.*

Proof. By Lemma 1, we have

$$(aa^*)^2 = a(a^* a)a^* = (a^* a)aa^* = a^*(a(aa^*))$$
$$= a^*(aa^*)a = a^* aa^* a = (a^* a)^2.$$

Hence, by the unicity of the square root, we have $aa^* = a^* a$.

Lemma 3. *If a is a dependent element of α, then $a^* a$ is a dependent element of α in Z.*

Proof. For any $z \in Z$, we have, by (1),

$$z^\alpha a^* a = a^* z^\alpha a = a^* az.$$

Hence $a^* a$ is a dependent element of $\alpha | Z$.

From Lemma 3 and a theorem of Nakamura and Takeda [3; Lemma 2], we can deduce that $a^* a$ is invariant under α: $(a^* a)^\alpha = a^* a$. However, we shall prove this by an elementary manner in the following

Lemma 4 (Nakamura-Takeda). *If A is abelian, and if a is a dependent element of an automorphism of A, then a is invariant:*

(3) $$a^\alpha = a.$$

Proof. At first, we shall show that a^α is also dependent if a is dependent. By (1), we have $a^\alpha x^\alpha = x^{\alpha^2} a^\alpha$, so that we have $a^\alpha y = y^\alpha a^\alpha$ for any $y \in A$, putting $y = x^\alpha$. The remainder of the proof is now a consequence of the following computation:

$$(a^\alpha - a)^*(a^\alpha - a) = a^{*\alpha} a^\alpha - a^* a^\alpha - a^{*\alpha} a + a^* a$$
$$= a^\alpha a^\alpha - a^* a^\alpha - aa^* + a^* a = 0.$$

Theorem 1. *If a is a dependent element of an automorphism α of a C^*-algebra A, then a is invariant under α.*

Proof. By Lemma 2 and (2), we have

$$a^* a = aa^* = a^* a^\alpha;$$

hence we have

(4) $$a^*(a^\alpha - a) = 0.$$

On the other hand, by (1), Lemmas 2, 3 and 4, we have

$$a^{*\alpha} a = aa^* = a^* a = (a^* a)^\alpha = a^{*\alpha} a^\alpha.$$

Hence we have

(5) $$a^{*\alpha}(a^\alpha - a) = 0.$$

Subtracting (4) from (5), we have

$$(a^{*\alpha} - a^*)(a^\alpha - a) = 0.$$

Therefore, we have (3).

Remark. Lemmas 1 and 2 are known long since when a is invertible, cf. [2; p. 15]. The full strength of Lemma 1 is observed at first by Kallman [3]. In the eyes of the specialists for seminormal operators, Lemma 1 states that a and a^* are quasinormal, so that Lemma 2 follows, cf. [4]. However, it seems to the authors that Lemma 2 is not recognized explicitly.

We wish to note that the results of this section are valid for C^*-algebras without the identity. Also, they are valid for suitably restricted Baer $*$-ring, since no metrical property is needed.

3. Applications. We shall apply the above results in some elementary special cases. At first, we shall call a C^*-algebra A is a *factorial* if the center Z of A consists of scalars.

Theorem 2. *In a factorial C^*-algebra, an automorphism is either inner or freely acting.*

Proof. If a is a nonzero dependent element of an automorphism α, then a^*a is central by Lemma 1, so that $a^*a=\lambda$ for some scalar $\lambda>0$ by the hypothesis. Since a is normal by Lemma 2, a is invertible, so that (1) implies

$$x^\alpha = axa^{-1}. \tag{6}$$

If $a=u|a|$ is the polar decomposition of a, then u is a unitary element of A and $|a|=\sqrt{\lambda}$, so that we have

$$x^\alpha = uxu^*, \tag{6'}$$

instead of (6), that is, inner automorphisms of a factorial are unitarily inner, cf. [2; p. 15] and [5; Lemma 1].

Theorem 3 (Nakamura-Takeda-Kallman). *In a factor, an automorphism α is outer if and only if α is freely acting.*

Proof. A factor is naturally a factorial, so that Theorem 3 follows from Theorem 2.

Remark. Theorem 3 is proved at first by Nakamura and Takeda [5; Lemma 1] for finite factors; their proof based on the fact that a finite factor is algebraically simple. They proved essentially that if an automorphism α of a simple unital C^*-algebra is outer then α is freely acting. Kallman [3] proved Theorem 3 in its generality based on his theorem which is given a proof in the below. Our proof is simpler and more elementary than theirs.

A completely analogous method gives the following generalization of a theorem of Kallman [3; Corollary 1.13]: We shall call an automorphism α *ergodic* if there is no element up to scalars which is invariant under α.

Theorem 4. *An ergodic automorphism of a nontrivial C^*-algebra is outer.*

Proof. If an ergodic automorphism α is inner satisfying (6), then

a is dependent for α, so that a is invariant under α by Theorem 1, which is clearly impossible by the ergodicity of α.

It is well-known that all powers of an ergodic measure preserving automorphism of a nonatomic probability space are freely acting. Choda [1] generalized this to every continuous von Neumann algebra. We shall give here, by virtue of Lemma 2, a partial converse of these theorems:

Theorem 5. *If α is an automorphism of a C^*-algebra and α^n is freely acting for some n, then α is freely acting.*

Proof. If α is not freely acting, then there is a nonzero dependent element a satisfying (1). Hence we have

$$a^n x = x^{\alpha^n} a^n,$$

for all $x \in A$. Since α^n is freely acting by the hypothesis, we have $a^n = 0$. Since a is normal by Lemma 2, we have $a = 0$, which is a contradiction.

If A is a unital C^*-algebra and B is a unital C^*-subalgebra of A, then a positive linear transformation ε of A onto B is called an *expectation* of A onto B in the sense of [6], cf. also [7], if

(7) $$(ab)^\varepsilon = a^\varepsilon b, \qquad (ba)^\varepsilon = ba^\varepsilon,$$

for every $a \in A$ and $b \in B$. An expectation ε is called *faithful* if $(a^*a)^\varepsilon = 0$ implies $a = 0$.

For an automorphism α of A, a (unital) C^*-subalgebra B is called *invariant* under α if $B = B^\alpha = \{x^\alpha ; x \in B\}$, and the set F of all invariant elements of α is called the *fixed subalgebra* for α.

Theorem 6. *Let α be an automorphism of a unital C^*-algebra A with the fixed subalgebra F. Suppose that there is a faithful expectation ε of A onto an invariant unital C^*-subalgebra B with $B \subset F^c$ where F^c is the relative commutant of F in A. If α is freely acting on B, then α is freely acting on A.*

Proof. If a is a dependent element of α, then $a \in F$ by Theorem 1. By (1), we have $a^*ax = a^*x^\alpha a$ for any $x \in A$, so that

$$a^*ab = a^*b^\alpha a = b^\alpha a^*a$$

for every $b \in B$ since $B \subset F^c$. Therefore, we have by (7)

$$(a^*a)^\varepsilon b = (a^*ab)^\varepsilon = (b^\alpha a^*a)^\varepsilon = b^\alpha (a^*a)^\varepsilon,$$

so that we have $(a^*a)^\varepsilon = 0$ by the hypothesis that α is freely acting on B. Hence $a = 0$ by the faithfulness of ε, and α is freely acting on A.

The following corollaries are now obvious by Theorem 6's proof:

Corollary 1. *If there is a faithful expectation of A onto F^c, then the free action of α on F^c implies the free action of α on A.*

Corollary 2. *Kallmann the free action of an automorphism α on Z implies the free action of α on A.*

Remark. The assumptions of the above Theorem are satisfied when the algebra A is a finite von Neumann algebra, by the theorem of Umegaki [7].

At this end, we shall generalize a theorem due to Nakamura and Takeda [5; Lemma 2] for C^*-algebras:

Theorem 7. *If A is an abelian C^*-algebra on which an automorphism α acts, then the set D of all dependent elements of α is an ideal in which every element is invariant under α.*

Proof. If $a, b \in D$ and $x, y \in A$, then we have

$$(a+b)x = ax + bx = x^\alpha a + x^\alpha b = x^\alpha(a+b)$$

and

$$(ya)x = yax = yx^\alpha a = x^\alpha ya;$$

hence D is an ideal. The closedness of D is obvious. The remainder of the theorem follows from Theorem 1.

4. A remark. If A is a von Neumann algebra on which an automorphism α acts, and if a is a dependent element of α, then there is a unitary operator $u \in A$ by Lemma 2 in the polar decomposition of a:

$$a = u|a|, \qquad |a| \in Z. \tag{8}$$

Hence, by (1), (8) and Lemma 1, we have

$$ux|a| = x^\alpha u|a|. \tag{9}$$

Therefore, by (9), we have the following theorem which is a key for the Kallman decomposition of automorphisms of von Neumann algebras:

Theorem 8 (Kallman). *If a is a dependent element of an automorphism α in a von Neumann algebra, then α is (unitarily) inner on the central carrier of a.*

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