124. *Commutative Semigroups of Type (m, p, n, q).* I

Monovariable Cases

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1. Introduction and notations.

The following problem has appeared in the study of generalized translations of commutative semigroups:

Problem. Let m, p, n, q be given positive integers. Find all commutative semigroups S satisfying the condition:

for every $x \in S$, there exists $y \in S$ such that

$$x^m y^p = x^n y^q.$$

The purpose of this and the subsequent notes is to give partial solutions to this problem.

We shall also consider some special classes $\boldsymbol{X}$ of commutative semigroups. They are archimedian semigroups, cancellative semigroups, and abelian groups. Every semigroup is assumed to be commutative and not empty. Also all groups are assumed to be abelian. The class of all semigroups is denoted by $\boldsymbol{S}$. The subclasses of $\boldsymbol{S}$ consisting of all archimedian semigroups, cancellative semigroups, groups are denoted by $\boldsymbol{A}, \boldsymbol{C}, \boldsymbol{G}$, respectively.

Let us denote by N the set of all positive integers, while $\bar{N}$ denotes the set of all non-negative integers: $\bar{N} = N \cup \{0\}$. Put $T = \bar{N} \times \bar{N} \times \bar{N} \times \bar{N}$.

After these definitions our revised problem can be stated as follows:

Problem. Let $(m, p, n, q) \in T$, and let $\boldsymbol{X}$ be any one of four classes $\boldsymbol{S}, \boldsymbol{A}, \boldsymbol{C}, \boldsymbol{G}$. Find all semigroups S belonging to the class $\boldsymbol{X}$ such that the following condition is satisfied:

($*$) For every $x \in S$ there exists $y \in S$ such that $x^m y^p = x^n y^q$ in S^1.

The symbol S^1 in the problem means the semigroup S with the identity element 1 adjoined, if S does not have the identity; otherwise, $S^1 = S$. For every element $x \in S$ we shall understand $x^0 = 1$ in S^1. For any terms not defined here, the reader should refer to [1].

A semigroup S is said to be of type (m, p, n, q), if S satisfies the condition ($*$) above. The class of all semigroups of type (m, p, n, q) is denoted by $\boldsymbol{S}(m, p, n, q)$, and put $\boldsymbol{X}(m, p, n, q) = \boldsymbol{X} \cap \boldsymbol{S}(m, p, n, q)$.

These classes define a quasi-order relation $\leq$ in T by $(m, p, n, q) \leq (m', p', n', q')$ if and only if $\boldsymbol{X}(m, p, n, q) \subseteq \boldsymbol{X}(m', p', n', q')$.

We shall use throughout these notes $\subseteq$ for the inclusion, and $\subset$ for the proper inclusion.

The quasi-order relation induces an equivalence relation $\sim$ in T and an order relation in the quotient set $T^*=T/\sim$. Actually, by taking a suitable representative from each equivalence class, the quotient T^* can be expressed isomorphically as an ordered subset $T^\#$ of T, and we may identify them if convenient. By this reason we shall use the same symbol $\leq$ for the order relation in T^* and in $T^\#$ and for the quasi-order relatisn in T. Whenever preferable, the class symbol X will be attached such as $T^*(X)$. $T^*(X)$ and $T^\#(X)$ are both called the type space for X. Similarly, the type spaces W^* and $W^\#$ may be defined for any subset W of T.

The subset of T consisting of all quadruples (m, p, n, q) such that some of the entries are restricted to be 0 but others are free will be studied in detail. And we shall denote by T_{hijk}, where h, i, j, k take 0 or 1, the subset of T defined by

$$T_{hijk}=N_h\times N_i\times N_j\times N_k, \text{ where } N_l=\begin{cases}\overline{N} & \text{if } l=1\\ \{0\} & \text{if } l=0.\end{cases}$$

Our objectives include the determination of the order structure of the type spaces $T^*(X)$. And to this end, we shall deal with various $T_{hijk}{}^*$ not only for auxiliary purpose but also for their own interest.

An element $\tau\in T$ is called a type. A type $\tau=(m, p, n, q)$ is called normal or normalized if either $n<m$ or $n=m$ and $p\leq q$.

Let us list here some of the immediate consequences from the definitions.

Lemma 1.1. $G\subset C\subset S$, $G\subset A\subset S$.

Lemma 1.2. $G(\tau)\subseteq C(\tau)\subseteq S(\tau)$, $G(\tau)\subseteq A(\tau)\subseteq S(\tau)$.

Lemma 1.3. $X(m, p, n, q)=X(n, q, m, p)$.

Corollary 1.4. *The type (m, p, n, q) is equivalent with the type (n, q, m, p). In other words, every type is equivalent with a normal type.*

Thus $T^\#(X)$ may be expressed as a subset of T containing only normal types.

Lemma 1.5. $X(m, p, m, p)=X$. *Especially* $X(0, 0, 0, 0)=X$.

We shall reserve the symbol ω for the normal type $(0, 0, 0, 0)$. Thus ω is the greatest element of the type space $T^\#(X)$.

In the present note, we shall consider the cases when only one variable appears in the equation explicitly, that is, either $m=n=0$ or $p=q=0$. The first case is rather trivial, and the result is stated here only for the completeness. So the main object in this note lies in the study of the second case.

2. Semigroups of type $(0, p, 0, q)$.

Denote by X^1 and X^i the classes of all semigroups belonging to the class $\boldsymbol{X}$ which have the identity and which have an idempotent, respectively. Clearly, we have

Lemma 2.1. $\boldsymbol{X}^1 \subseteq \boldsymbol{X}^i \subseteq \boldsymbol{X}$.

Remark. The inclusions are proper for $\boldsymbol{X}=\boldsymbol{S}$ and for $\boldsymbol{X}=\boldsymbol{A}$. However, note that

$$\boldsymbol{C}^1=\boldsymbol{C}^i \subset \boldsymbol{C}.$$

Also

$$\boldsymbol{G}^1=\boldsymbol{G}^i=\boldsymbol{G}=\boldsymbol{A}^1.$$

Theorem 2.2. *Let $p,\ q \in \bar{N}$. Then*

$$\boldsymbol{X}(0,p,0,q)=\begin{cases}\boldsymbol{X} & \text{if } p=q\\ \boldsymbol{X}^i & \text{if } p\neq q,\ pq\neq 0\\ \boldsymbol{X}^1 & \text{if } p\neq q,\ pq\neq 0.\end{cases}$$

Corollary 2.3. *Let $p,\ q \in \bar{N}$. Then*

$$\boldsymbol{S}(0,p,0,q)=\begin{cases}\boldsymbol{S} & \text{if } p=q\\ \boldsymbol{S}^i & \text{if } p\neq q,\ pq\neq 0\\ \boldsymbol{S}^1 & \text{if } p\neq q,\ pq=0.\end{cases}$$

$$\boldsymbol{A}(0,p,0,q)=\begin{cases}\boldsymbol{A} & \text{if } p=q\\ \boldsymbol{A}^i & \text{if } p\neq q,\ pq\neq 0\\ \boldsymbol{G} & \text{if } p\neq q,\ pq=0.\end{cases}$$

$$\boldsymbol{C}(0,p,0,q)=\begin{cases}\boldsymbol{C} & \text{if } p=q\\ \boldsymbol{C}^1 & \text{if } p\neq q.\end{cases}$$

$$\boldsymbol{G}(0,p,0,q)=\boldsymbol{G}.$$

Theorem 2.4. *$T_{0101}{}^*(\boldsymbol{X})$ is a chain consisting of three elements for $\boldsymbol{S}$ and for $\boldsymbol{A}$, two elements for $\boldsymbol{C}$, and a single element for $\boldsymbol{G}$, respectively.*

We may take for representatives the normal types $\alpha=(0,0,0,1)$, $\beta=(0,1,0,2)$, and ω. Thus we have

Corollary 2.5. *Let $U=T_{0101}$. Then*

$$U^*(\boldsymbol{S})=U^*(\boldsymbol{A})=\{\alpha,\ \beta,\ \omega\},\ \alpha<\beta<\omega.$$
$$U^*(\boldsymbol{C})=\{\alpha,\ \omega\},\ \alpha<\omega.$$
$$U^*(\boldsymbol{G})=\{\omega\}.$$

Remark. The choice of the representatives can be arbitrary. For example, we may take the representatives $\tau_i=(0,1,0,i)$, $i=0,1,2$. Then $\tau_0\sim\alpha$, $\tau_2=\beta$ and $\tau_1\sim\omega$. Note that τ_0 is not a normal type.

3. The semilattice L^1.

For the latter purpose let us introduce here the semilattice L^1.

Let L be the set of all sequence $n=(n_1, n_2, n_3, \cdots)$ of all non-negative integers $n_k \in \bar{N}$ such that $n_k=0$ for all but finitely many k. Call n_k the k-th coordinate of $\boldsymbol{n}$. Define the coordinatewise ordering $\leq$ in L. Thus $\boldsymbol{m}\leq\boldsymbol{n}$ if and only if $m_k\leq n_k$ for all $k \in N$. Then L forms not only an ordered set but also a semilattice, where the operation $\wedge$ is the meet operation for each component:

$m \wedge n = (l_1, l_2, \cdots)$ if and only if $l_k = \mathrm{Min}(m_k, n_k)$ for all k.

By shifting the coordinate by one step we have

Lemma 3.1. $L \cong \bar{N} \times L$.

Let p_k be the k-th prime. Then every positive integer $n \in N$ has the unique canonical factorization $n = \prod_{k \in N} p_k^{n_k}$. Put $f(n) = (n_1, n_2, \cdots)$. Then f is a bijection from N onto L. Moreover, this correspondence is order preserving, where N is endowed with the ordering by divisibility.

Lemma 3.2. $L \cong (N, |)$.

Remark. The isomorphisms in the above two lemmas are not only order isomorphisms but also semilattice isomorphisms. Also note that the semilattice L does not have the greatest element. By adjoining the greatest element to L, we have L^1, which is not only semilattice but also a complete lattice.

Lemma 3.3. $L^1 \cong (\bar{N}, |) \cong (\bar{N} \times L)^1$.

Note that 0 is the smallest element in $\bar{N}$, while it is the greatest element in $(\bar{N}, |)$.

4. Semigroups of type $(m, 0, 0, 0)$.

In the rest of the present paper, we shall deal with the study of the semigroups of type $(m, 0, n, 0)$.

Let $V = T_{1010}$, i.e., $V = \{(m, 0, n, 0) \in T \mid m, n \in \bar{N}\}$.

Define subsets V_0 and V_1 of V by

$$V_0 = \{(m, 0, 0, 0) \mid m \in \bar{N}\} = T_{1000}.$$

$$V_1 = \{(m, 0, n, 0) \mid 0 < n < m\}.$$

Then V_0 and V_1 are disjoint and the union

$$V' = V_0 \cup V_1$$

contains only normal types in V. Moreover, every type in V is equivalent with a type in V'. Thus we can take $V^{\#}(X)$ as a subset of V' for each X.

Let $m \in \bar{N}$. Then a group G is said to be of class m, if $x^m = 1$ for every element $x \in G$. The class of all groups of class m is denoted by $\boldsymbol{G}(m)$. Thus $\boldsymbol{G}(1)$ consists of all singleton groups, while $\boldsymbol{G}(0) = \boldsymbol{G}$.

Lemma 4.1. $\boldsymbol{G}(m) \subseteq \boldsymbol{G}(n)$ *if and only if* $m \mid n$.
$\boldsymbol{G}(m) = \boldsymbol{G}(n)$ *if and only if* $m = n$.
$\boldsymbol{G}(m) \cap \boldsymbol{G}(n) = \boldsymbol{G}(d)$,
where $d = GCD(m, n)$, *the greatest common divisor of* m *and* n.

Lemma 4.2. *Let* $m \in \bar{N}$. *Then*

$$\boldsymbol{X}(m, 0, 0, 0) = \begin{cases} \boldsymbol{G}(m) & \text{if} \quad m \neq 0 \\ \boldsymbol{X} & \text{if} \quad m = 0. \end{cases}$$

Therefore the type space V_0^* can be identified with $(\bar{N}, |)$, and by Lemma 3.3 we have

Theorem 4.3. *The type space* $T_{1000}^*(\boldsymbol{X})$ *is isomorphic with* L^1 *for all four cases for* $\boldsymbol{X}$.

5. Groups and cancellative semigroups of type $(m, 0, n, 0)$.

Theorem 5.1. *Let $0 \leq n \leq m$. Then*

$$G(m, 0, n, 0) = G(m-n, 0, 0, 0) = G(m-n)$$

$$C(m, 0, n, 0) = C(m-n, 0, 0, 0) = \begin{cases} G(m-n) & \text{if } n<m \\ C & \text{if } n=m. \end{cases}$$

Therefore for groups and for cancellative semigroups the function $f: V' \to V_0$ defined by $f(m, 0, n, 0) = (m-n, 0, 0, 0)$ is a choice function of representatives.

Corollary 5.2. $V^{\#}(G) = V^{\#}(C) = V_0$.

Theorem 5.3. *The type spaces $V^*(G)$ and $V^*(C)$ for groups and for cancellative semigroups are isomorphic with the complete lattice L^1, where $V = T_{1010}$. Moreover, the meet operation in L^1 corresponds with the intersection of the classes determined by the elements of V^*.*

6. Archimedean semigroups of type $(m, 0, n, 0)$.

For a semigroup S, the intersection M of all ideals of S is either empty or else an ideal of S. If M is not empty, then it is the unique minimal ideal of S and it is necessarily a group. Let X^m denote the class of all semigroups belonging to the class X which have the minimal ideal. Thus

$$G = G^m = C^m \subset A^m \subset S^m.$$

Let $n, r \in N$. Denote by $X(n; r)$ the class of all semigroups $S \in X^m$ such that

(1) the minimal ideal M of S is a group of class r: $M \in G(r)$.

(2) $x^n \in M$ for every $x \in S$.

Obviously,

$$S(1; r) = G(r).$$

Lemma 6.1. *An archimedean semigroup A with an idempotent e has the minimal ideal $M = eA$, and for every $x \in A$ there exists $n \in N$ such that $x^n \in M$. Conversely, if S is a semigroup with a minimal ideal M such that every element of S has some power in M, then S is an archimedean semigroup with an idempotent.*

Lemma 6.2. $A(n; r) = S(n; r)$ *for all* $n, r \in N$.

Lemma 6.3. *Let $n, n', n'', r, r', r'' \in N$. Then*

$A(n; r) \subseteq A(n'; r')$ *if and only if* $n \leq n'$ *and* $r \mid r'$.

$A(n; r) = A(n'; r')$ *if and only if* $n = n'$ *and* $r = r'$.

$A(n; r) \cap A(n'; r') = A(n''; r'')$, *if* $n'' = \mathrm{Min}\,(n, n')$ *and* $r'' = GCD(r, r')$.

Remark. This lemma does also hold for S because of Lemma 6.2.

Theorem 6.4. *Let $0 \leq n < m$. Then*

$$A(m, 0, n, 0) = \begin{cases} A(n; m-n), & \text{if } n \neq 0 \\ A(1; m) & \text{if } n = 0. \end{cases}$$

Especially

Corollary 6.5. $A(m, 0, 0, 0) = A(m+1, 0, 1, 0)$ for all $m \in \bar{N}$.

It follows immediately that every type in V_0, except ω, is equivalent with a type in V_1. Therefore, we can take the type space $V^{\#}(A)$ as a subset of $V_1 \cup \{\omega\}$. Thus by Lemma 6.3 and Theorem 6.4, we have

Lemma 6.6. $V^{\#}(A) = V_1 \cup \{\omega\}$.

Let $f: V_1 \to \bar{N} \times N$ be defined by $f(m, 0, n, 0) = (n-1, m-n)$. Then it follows that f is a bijection and it is order preserving by regarding the range as the ordered set $(\bar{N}, \leq) \times (N, |)$, which is isomorphic with $\bar{N} \times L$ by Lemma 3.2. Therefore $V_1(A) \cong \bar{N} \times L$. Thus by Lemma 3.3 we have $V^{\#}(A) = V_1 \cup \{\omega\} \cong (\bar{N} \times L)^1 \cong L^1$.

Theorem 6.7. *The type space $T^*_{1010}(A)$ is isomorphic with the complete lattice L^1. Moreover, the meet operation in L^1 corresponds with the class intersection.*

7. Semigroups of type $(m, 0, n, 0)$.

Lemma 7.1. Every semigroup is a semilattice of archimedian semigroups. More precisely, if S is a semigroup, then there exists a semilattice J and for each $i \in J$ there corresponds a subsemigroup S_i of S such that

(1) S_i is archimedian for each $i \in J$.

(2) S_i are disjoint to each other and S is the union of them.

(3) $S_i S_j \subseteq S_{ij}$ for every $i, j \in J$.

Moreover, each S_i is a maximal archimedian subsemigroup of S, and J is uniquely determined up to isomorphism. In fact, it is the greatest semilattice quotient of S ([1] pp. 130–132).

We shall call S_i an archimedian component of S.

Lemma 7.2. *A semigroup S is of type $(m, 0, n, 0)$ if and only if every archimedian component of S is of type $(m, 0, n, 0)$.*

Therefore, the problem on the type space of semigroups is completely reduced to the corresponding problem for archimedian semigroups. Thus we can take $V^{\#}(S) = V^{\#}(A)$.

Theorem 7.3. *The type space $T^*_{1010}(S)$ is isomorphic with the complete lattice L^1. Moreover, the meet operation in L^1 corresponds with the class intersection.*

Reference

[1] A. H. Clifford and G. B. Preston: The algebraic theory of semigroups, Vol. 1 (1964). Amer. Math Soc.