

123. On Power Cancellative Archimedean Semigroups

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(Comm. by Kenjiro SHODA, M. J. A., Sept. 12, 1972)

The following problem was raised by Tamura*¹

Problem. Is a power cancellative archimedean semigroup necessarily cancellative?

The purpose of this short note is to give an affirmative answer to this problem. Throughout the paper a semigroup means a commutative semigroup, and a subsemigroup and an ideal are assumed non-empty. A semigroup is archimedean if for any two elements x, y , some power of x is divisible by y . Any terminology not defined here should be referred to [1].

Lemma 1. *Every ideal of an archimedean semigroup is an absorbing subset, in the sense that, every element has some power contained in the ideal.*

Proof. Let A be an archimedean semigroup and B an ideal of A . Let $b \in B$. Since A is archimedean, for every element $a \in A$ there exist a positive integer n and an element $c \in A$ such that $a^n = bc$. But $bc \in B$, because B is an ideal. Hence it follows that $a^n \in B$.

Remark 1. The condition in the lemma is actually the necessary and sufficient condition for a semigroup to be archimedean.

A semigroup is called power cancellative, if $x^n = y^n$ for some positive integer n always implies $x = y$.

Theorem 1. *Every power cancellative archimedean semigroup is cancellative.*

Proof. Let A be a power cancellative archimedean semigroup. Assume that $ac = bc$, where $a, b, c \in A$. Consider the subset C of A defined by $C = \{x \in A \mid ax = bx\}$. Then C is not empty, because $c \in C$, and it is easily seen that C is an ideal of A . Hence it is absorbing by Lemma 1. Therefore there exist positive integers m and n such that $a^m, b^n \in C$. From the definition of C , it follows immediately that $a^k x = b^k x$ for every element $x \in C$ and for every positive integer k .

Therefore we have

$$a^{m+n} = a^m b^n = b^{m+n}.$$

Since A is power cancellative, it follows that $a = b$.

This completes the proof of the theorem.

Remark 2. A cancellative archimedean semigroup may not be

*¹) In his letter to one of the authors.

power cancellative. In other words, power cancellativity is stronger than cancellativity for archimedean semigroups. But this fails for semigroups. That is, there are power cancellative semigroups which are not cancellative. The following examples will explain the situation:

Example 1. Let T be an abelian torsion group containing more than one element. Let N be the additive semigroup of positive integers. Then both T and the direct product $N \times T$ are cancellative archimedean semigroups, but neither is power cancellative. Note that the former has an idempotent, but the latter does not.

Example 2. Let L be a semilattice containing more than one element. Then the direct product $N \times L$ is power cancellative, but not cancellative. Note that this semigroup is not archimedean.

Lemma 2. *A semigroup is a semilattice of archimedean subsemigroups.*

Proof. See [1].

Theorem 2. *Every power cancellative semigroup is a semilattice of cancellative archimedean subsemigroups.*

Proof. Obvious by Theorem 1 and Lemma 2.

Remark 3. The converse of the above theorem is not true in general.

Reference

- [1] A. H. Clifford and G. B. Preston: *The Algebraic Theory of Semigroups*, Vol. 1. Math. Surveys, No. 7, Amer. Math. Soc. (1964).