122. *Remarks on the Asymptotic Behavior of the Solutions of Certain Non-Autonomous Differential Equations*

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1. Introduction. In this paper we consider the asymptotic behavior of the solutions of the non-autonomous, nonlinear differential equation;

(1.1) $$\dot{x}=A(t)x+f(t,x)$$

where x, f are n-dimensional vectors, $A(t)$ is a bounded continuously differentiable $n\times n$ matrix for $t\geqq 0$, and $f(t,x)$ is a continuous in (t,x) for $t\geqq 0$, $\|x\|<\infty$, here $\|\cdot\|$ denotes an Euclidean norm. And consider

(1.2) $$\dddot{x}+a(t)\ddot{x}+b(t)g(x,\dot{x})\dot{x}+c(t)h(x)=p(t,x,\dot{x},\ddot{x})$$

where $a(t), b(t), c(t)$ are positive, continuously differentiable and g, h, p are continuous real-valued functions depending only on the arguments shown, the dots indicate the differentiation with respect to t. In this note, certain conditions are obtained under which all solutions of (1.1) tend to zero as $t\to\infty$.

In [6], the author studied the asymptotic behavior of the solution of the equation

(1.3) $$\dddot{x}+a(t)f(x,\dot{x})\ddot{x}+b(t)g(x,\dot{x})\dot{x}+c(t)h(x)=e(t)$$

under the assumptions that $|a'(t), |b'(t)|, |c'(t)|$ and $e(t)$ are integrable and suitable conditions on f, g, h. Here we assume the condition that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\{|a'(s)|+|b'(s)|+|c'(s)|\}ds$$

has an infinitesimal upper bound,

to prove the every solution of (1.2) tends to zero as $t\to\infty$. Conditions on $p(t,x,y,z)$ are also relaxed. Theorem 2 generalizes the Ezeilo's result [5] in which he considered the equation

(1.4) $$\dddot{x}+a_1\ddot{x}+a_2\dot{x}+f_3(x)=p_1(t,x,\dot{x},\ddot{x}),$$

where a_1, a_2 are positive constants.

The main tool used in this work is Lemma 1 which is a specialization of the result obtained by F. Brauer [1]. Using this Lemma and Liapunov functions, we shall obtain Theorem 1 and Theorem 2. Lemma 1 is especially convenient to study the non-autonomous differential equations.

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2. Main lemma. Consider a system of differential equations

(2.1) $$\dot{x}=F(t,x),$$

where x and F are n-dimensional vectors.

Lemma 1. *Suppose that $F(t,x)$ of (2.1) is continuous in $I\times R^n$ ($I=[0,\infty)$) and that there exists a Liapunov function $V(t,x)$ defined in $I\times R^n$ satisfying the following conditions;*

(i) *$a(\|x\|)\leqq V(t,x)\leqq b(\|x\|)$, where $a(r)\in CIP$ (i.e. continuous and increasing positive definite functions), $a(r)\to\infty$ as $r\to\infty$ and $b(r)\in CIP$,*

(ii) *$\dot{V}_{(2.1)}(t,x)\leqq -cV(t,x)+\lambda_1(t)V(t,x)+\lambda_2(t)\phi(V(t,x))$, where $c>0$ is a constant and $\lambda_i(t)\geqq 0(i=1,2)$ are continuous functions satisfying*

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\lambda_1(s)ds<c,\quad \int_t^{t+1}\lambda_2(s)ds\to 0 \text{ as } t\to\infty,$$

and $\phi(u)$ is a continuous, non-negative function for $u\geqq 0$ satisfying $\phi(u)=0(u)$ as $u\to\infty$.

Then, all solutions $x(t)$ of (2.1) are uniform-bounded and satisfy $x(t)\to 0$ as $t\to\infty$.

The detailed proof of Lemma 1 is to appear in some journal.

3. Theorems. *Let $A(t)$ satisfy the condition (i) of the following Theorem 1, and $P(t)$ be a solution of the matrix equation*

(3.1) $$A^T(t)P(t)+P(t)A(t)=-I.$$

Notice that $P(t)$ is bounded for bounded $A(t)$. The following propositions are due to J. R. Dickerson [2].

Proposition A. *$x^TP(t)x\geqq C\|x\|^2$, where C is a positive constant.*

Proposition B. *$|x^T\dot{P}(t)x|\leqq 2\|\dot{A}(t)\|\cdot\|P(t)\|x^TP(t)x$, where $\dot{P}(t)$ and $\dot{A}(t)$ denote the time derivative of matrices $P(t)$ and $A(t)$ respectively.*

Theorem 1. *Suppose that the following conditions are satisfied;*

(i) *the eigenvalues of $A(t)$ have negative real parts strictly bounded away from zero for all $t\geqq 0$,*

(ii) $$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\|\dot{A}(s)\|ds<\frac{1}{2P_1^2}$$
where $P_1=\sup_{t\geqq 0}\|P(t)\|$,

(iii) $\|f(t,x)\|\leqq\gamma_1(t)+\gamma_2(t)\|x\|^\rho$
where $\gamma_1(t),\gamma_2(t)$ are non-negative, continuous for $t\geqq 0$ and ρ is a constant such that $0\leqq\rho\leqq 1$,

(iv) $$\int_t^{t+1}\gamma_i(s)ds\to 0 \text{ as } t\to\infty\ (i=1,2).$$

Then, all solutions $x(t)$ of (1.1) are uniform-bounded and satisfy $x(t)\to 0$ as $t\to\infty$.

Next, we consider the equation (1.2) and assume that $g(x,y)$ and $g_x(x,y)$ are continuous, real-valued for all (x,y) and $h(x)$ is continuously differentiable for all x.

Theorem 2. *Suppose that $a(t), b(t)$ and $c(t)$ are continuously differentiable functions, and the following conditions are satisfied;*

(i) $A \geqq a(t) \geqq a_0 > 0, B \geqq b(t) \geqq b_0 > 0, C \geqq c(t) \geqq c_0 > 0,$ *for* $t \in I = [0, \infty)$,

(ii) $h(0)=0, \dfrac{h(x)}{x} \geqq \delta > 0 \quad (x \neq 0),$

(iii) $0 < g_0 \leqq g(x, y) \leqq g_0 + \dfrac{4\delta c_0}{B b_0 g_0}, \quad y g_x(x, y) \leqq 0$ *for all* $(x, y) \in R^2$,

(iv) $\dfrac{a_0 b_0 g_0}{C} > h_1 \geqq h'(x),$

(v) $\limsup_{t\to\infty} \dfrac{1}{t}\displaystyle\int_0^t \{|a'(s)|+|b'(s)|+|c'(s)|\} ds$ *has an infinitesimal upper bound,*

(vi) $|p(t, x, y, z)| \leqq p_1(t) + p_2(t)(x^2+y^2+z^2)^{\rho/2} + \Delta_1 (x^2+y^2+z^2)^{1/2}$ *where* ρ, Δ_1 *are constants such that* $0 \leqq \rho \leqq 1, \Delta_1 \geqq 0$ *and* $p_1(t), p_2(t)$ *are non-negative, continuous functions,*

(vii) $\displaystyle\int_t^{t+1} p_i(s) ds \to 0$ *as* $t \to \infty$ $(i=1, 2)$.

If Δ_1 *is sufficiently small, then every solution* $x(t)$ *of* (1.2) *is uniform-bounded and satisfies* $x(t) \to 0, \dot{x}(t) \to 0, \ddot{x}(t) \to 0$ *as* $t \to \infty$.

4. Proof of Theorems. For the proof of Theorem 1, we consider the Liapunov function

(4.1) $$U(t, x) = x^T P(t) x.$$

By virture of Proposition A and the boundedness of $P(t)$, we have

$$C\|x\|^2 \leqq V(t, x) \leqq P_1 \|x\|^2.$$

A simple calculation shows that

$$\dot{V}_{(1.1)}(t, x) \leqq -\frac{1}{P_1} V(t, x) + 2P_1 \|\dot{A}(t)\| V(t, x) + 2\{\gamma_1(t) + \gamma_2(t)\}\left\{\left(\frac{V(t, x)}{C}\right)^{1/2} + \left(\frac{V(t, x)}{C}\right)^{(1+\rho)/2}\right\}.$$

Hence, the assumptions of Lemma 1 hold and the proof of Theorem 1 is completed.

For the proof of Theorem 2, the following Liapunov function is constructed;

(4.2) $$V(t, x, y, z) = V_0(t, x, y, z) + V_1(t, x, y, z)$$

where V_0 and V_1 are defined by

(4.3) $$2\mu_1 V_0 = 2\mu_1 c(t) \int_0^x h(\xi) d\xi + 2c(t) h(x) y + 2b(t) \int_0^y g(x, \eta) \eta d\eta + \mu_1 a(t) y^2 + 2\mu_1 y z + z^2,$$

(4.4) $$2V_1 = \mu_2 g_0 b(t) x^2 + 2a(t) c(t) \int_0^x h(\xi) d\xi + [a^2(t) - \mu_2] y^2 + 2b(t) \int_0^y g(x, \eta) \eta d\eta + z^2 + 2\mu_2 a(t) x y + 2\mu_2 x z + 2a(t) y z + 2c(t) h(x) y$$

and

$$\frac{Ch_1}{b_0 g_0} < \mu_1 < a_0, \; 0 < \mu_2 < \frac{a_0 b_0 g_0 - Ch_1}{A}.$$

A good calculation shows that the above Liapunov function satisfies the hypotheses of Lemma 1.

The detailed proof of Theorem 2 is to appear in some journal with the proof of Lemma 1.

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