171. *On the Regularity of Domains for the First Boundary Value Problem for Semi-linear Parabolic Partial Differential Equations*

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In this short note, we shall prove that a domain $D \subset R^{n+1}$ is regular for the first boundary value problem (=the Dirichlet problem or the initial-boundary value problem) for the semi-linear parabolic partial differential equation:

$$\text{(E)} \qquad \boldsymbol{P}u \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = f\left(x, t, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right),$$

if it is regular for $\boldsymbol{P}u \leqq -1$.

It is well known that even for the simplest equation of this kind, namely, for the heat equation

$$\text{(H)} \qquad \bigtriangledown u \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} = 0,$$

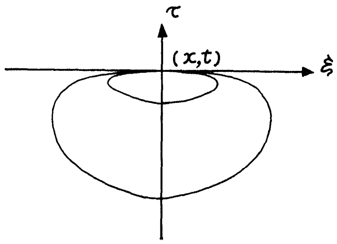
there may not be a solution u for the first boundary value problem if we require u to take values prescribed on the (whole) topological boundary of the domain. For example, consider the first boundary value problem for (H) for $n=1$ for the domain $\{(x, t); 0<x<1, 0<t<1\}$. Values of the solution $u(x, t)$ on the upper boundary $\{(x, t); 0 \leqq x \leqq 1, t=1\}$ are determined by the values of u given on the side boundary $\{(x, t); x=0 \text{ or } 1, 0 \leqq t \leqq 1\}$ and the lower boundary $\{(x,t); 0 \leqq x \leqq 1, t=0\}$.

Prompted by this example, let us split the topological boundary ∂D of a domain D bounded by a finite number of sufficiently smooth hypersurfaces into three parts, namely, i) Side boundary $\partial_s D$: closure of the part where the outer normal is not parallel to the time axis, ii) Lower boundary $\partial_l D$: closure of the part where the outer normal is in the $-t$ direction, and iii) Upper boundary $\partial_u D$: interior of the part where the outer normal is in $+t$ direction. We shall call the set $\partial_p D \equiv \partial_l D \cup \partial_s D$ the *parabolic boundary* of D, which is the set where we should give the boundary data. In other words, a point of $\partial_u D$ must be considered parabolically an interior point of D. So, the question to be asked will be: is there always a solution of $\bigtriangledown u=0$ (or more generally, $\boldsymbol{P}u=f$) in D admitting a continuous boundary value prescribed on $\partial_p D$?

Another example shows that there is not always a solution. Let $C(P, r)$ be the parabolic circle (sphere) for the heat equation (H) for

$n=1$ with centre $P=(x,t)$ and radius r. By this we mean that $C(P,r)$ is the curve expressed by a parameter θ as follows:

$$\begin{cases}\xi = x+\sqrt{2}\,r\sin\theta\sqrt{\log\operatorname{cosec}^2\theta} \\ \tau = t-r^2\sin^2\theta, \ -\dfrac{\pi}{2}\leqq\theta\leqq\dfrac{\pi}{2}.\end{cases}$$



Note that the curve $C(P,r)$ is the level curve $E(x,t;\xi,\tau)=1/r$ of the elementary solution

$$E(x,t;\xi,\tau)=\begin{cases}\dfrac{1}{\sqrt{t-\tau}}\exp\left(-\dfrac{(x-\xi)^2}{4(t-\tau)}\right) & \tau<t \\ 0 & \tau\geqq t.\end{cases}$$

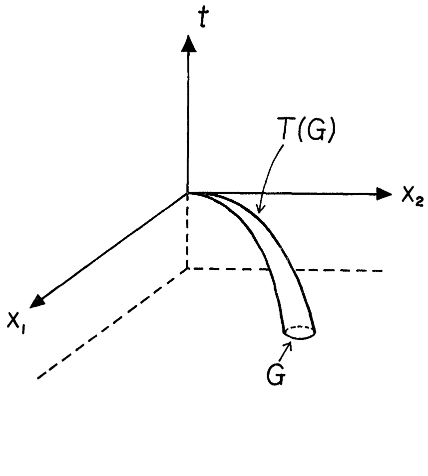
If u satisfies the heat equation, we see by Green's formula that

$$\int_{-\pi/2}^{\pi/2} u(\xi,\tau)\cos\theta\sqrt{\log\operatorname{cosec}^2\theta}\,d\theta=u(x,t).$$

This mean value theorem shows that the top point P of $C(P,r)$ is an irregular point for the domain surrounded by $C(P,r)$, for if we give continuous boundary data β on $C(P,r)$ that vanishes except in a small neighbourhood of P, where we assume $\beta>0$ with $\beta(P)=1$, then the solution u admitting β on $C(P,r)$, if it *did* exist, must satisfy

$$1=\beta(P)=u(x,t)=\int_{-\pi/2}^{\pi/2}\beta(\xi,\tau)\cos\theta\sqrt{\log\operatorname{cosec}^2\theta}\,d\theta<1.$$

Recently, E. G. Effros and J. L. Kazdan [1] proved that a boundary point where the outer normal is in the $+t$ direction is regular for the heat equation if this point is parabolically touchable. They defined the parabolical touchability as follows. If $u(x,t)$ is a solution of the



heat equation (H) then clearly $u(2^\alpha x, 4^\alpha t)$ is also a solution. By letting $\tau(\alpha)(x,t)=(2^\alpha x, 4^\alpha t)$, and $T(G)=\{\tau(\alpha)(x,-1); (x,-1)\in G, -\infty<\alpha<\infty\} \cup \{(0,0)\}$ where G is a closed n-sphere in the hyperplane $\{(x,t); x\in R^n, t=-1\}$, they defined that a point Q of $\partial_p D$ is parabolically touchable if, upon translating D so that $Q=(0,0)$, there is a tusk $T(G)$ with $T(G)\cap \bar{D}=\{(0,0)\}$. This tusk plays a role analogous to that of Poincaré's cone for the Laplace equation.

Thus, by the following theorem, which asserts that a domain is regular for (E) if it is regular for $Pu\leqq -1$, we see that a domain D is regular for

$$\bigtriangledown u=f\left(x,t,u,\frac{\partial u}{\partial x_1},\cdots,\frac{\partial u}{\partial x_n}\right),$$

if each point of the parabolic boundary $\partial_p D=\partial_l D\cup\partial_s D$ of D is parabolically touchable.

In the sequel, we use the following notations. For a function $f(x)$ defined on a set A, $f^*(y)$ for $y\in\bar{A}$ denotes $\lim_{x\to y}\sup_{x\in D} f(x)$ and $f_*(y)$ denotes $\lim_{x\to y}\inf_{x\in D} f(x)$. We denote by $\mathcal{K}(D)$ the set of functions defined on a domain $D\subset R^{n+1}$ which are twice continuously differentiable in x and once continuously differentiable in t.

Theorem. *We consider the equation*

$$\text{(E)}\qquad \boldsymbol{P}u\equiv\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i\partial x_j}-\frac{\partial u}{\partial t}=f\left(x,t,u,\frac{\partial u}{\partial x_1},\cdots,\frac{\partial u}{\partial x_n}\right),$$

where $\{a_{ij}(x,t)\}$, each $a_{ij}(x,t)$ being assumed to be bounded on a bounded domain $D\subset R^{n+1}$, is symmetric and positive definite, and f is a function satisfying the following condition: for any $M>0$ there exist B and Γ such that

$$|f(x,t,u,p_1,\cdots,p_n)|\leqq B\sum_{i,j=1}^n a_{ij}(x,t)p_ip_j+\Gamma$$

for $P=(x,t)\in D$, $|u|\leqq M$, $p=(p_1,\cdots,p_n)\in R^n$. Let $P_0=(x_0,t_0)$ be a point of $\partial_p D$. Assume that there exists a function $\psi(x,t)\in\mathcal{K}(D)$ such that $\psi_*(P)\geqq 0$ for $P\in\partial_p D$, $\lim_{D\ni P\to P_0}\psi(P)=0$, and $\boldsymbol{P}\psi\leqq -1$ on D[1]. *Let $\beta(x,t)$ be a bounded function on $\partial_p D$. Then, for any $\varepsilon>0$, $K<0$ and $L>0$, there exist a neighbourhood $V=V(\beta,\varepsilon,K,L)$ and barrier functions $\bar{\omega}(x,t)$, $\underline{\omega}(x,t)\in\mathcal{K}(D\cap V)$ satisfying*

(i) $\bar{\omega}^*(P_0)<\beta^*(P_0)+\varepsilon$, $\quad\underline{\omega}_*(P_0)>\beta_*(P_0)-\varepsilon$,

(ii) $\bar{\omega}_*(P)>\beta(P)$, $\quad\underline{\omega}_*(P)<\beta(P)\quad$ *for* $P\in\bar{V}\cap\partial_p D$,

(iii) $\bar{\omega}_*(P)>L$, $\quad\underline{\omega}^*(P)<K\quad$ *for* $P\in D\cap\partial V$,

$$\text{(iv)}\quad\begin{cases}\boldsymbol{P}\,\bar{\omega}(x,t)<f\left(x,t,\bar{\omega}(x,t),\dfrac{\partial\bar{\omega}}{\partial x_1},\cdots,\dfrac{\partial\bar{\omega}}{\partial x_n}\right)\\[2ex] \boldsymbol{P}\,\underline{\omega}(x,t)>f\left(x,t,\underline{\omega}(x,t),\dfrac{\partial\underline{\omega}}{\partial x_1},\cdots,\dfrac{\partial\underline{\omega}}{\partial x_n}\right)\qquad \textit{on } D.\end{cases}$$

1) This assumption implies $\psi(x,t)\geqq 0$ on D. See [2], p. 533, [4], p. 12.

Proof. Since K, L are arbitrary constants, we may assume that $-K=L>\sup\{|\beta(x,t)|;(x,t)\in\partial_p D\}$. Let $C=2L=L-K$. Then by the assumption of the theorem, there exist $B=B(M)$ and $\Gamma=\Gamma(M)$ such that

$$|f(x,t,u,p_1,\cdots,p_n)|\leqq B\sum a_{ij}(x,t)p_ip_j+\Gamma$$

for $(x,t)\in D, |u|\leqq M, -\infty<p_i<\infty$ $(i=1,\cdots,n)$. Since

$$\beta^*(P_0)=\limsup_{\partial_pD\ni P\to P_0}\beta(P),\qquad \beta_*(P_0)=\liminf_{\partial_pD\ni P\to P_0}\beta(P),$$

there exists a cylindrical neighbourhood $\{(x,t);|x-x_0|<\delta,|t-t_0|<\eta\}$ of P_0 such that $\beta_*(P_0)-\varepsilon/2<\beta(P)<\beta^*(P_0)+\varepsilon/2$ for $P=(x,t)\in\partial_pD$ in this neighbourhood. Letting

$$\phi(x)=\frac{1}{C_1}\{\exp(CC_1|x-x_0|^2/\delta^2)-1\},$$

we set

$$\bar{\omega}(x,t)=\frac{1}{C_1}\log[C_1\{N\psi(x,t)+\phi(x)\}+1]+C_2|t-t_0|^2+\beta^*(P_0)+\frac{\varepsilon}{2},$$

$$\underline{\omega}(x,t)=\frac{-1}{C_1}\log[C_1\{N\psi(x,t)+\phi(x)\}+1]-C_2|t-t_0|^2+\beta_*(P_0)-\frac{\varepsilon}{2},$$

where C_1, C_2 and N are constants to be determined later.

Since $\lim_{D\ni P\to P_0}\psi(P)=0$ and $\lim_{D\ni P\to P_0}\phi(x)=0$, we have

$$\lim_{D\ni P\to P_0}\bar{\omega}(P)=\beta^*(P_0)+\frac{\varepsilon}{2},$$

which shows that $\bar{\omega}(P)$ satisfies (i).

Let $U=\{(x,t);|x-x_0|<\delta,|t-t_0|<\eta\}$, and $S=\{(x,t)\in U\cap D; N\psi(x,t)+\phi(x)\geqq(1/C_1)(\exp CC_1-1)\}$. Then $P_0=(x_0,t_0)$ is not in S. Let V be the largest open neighbourhood of P_0 in U such that $V\subset U-S$.

For $P=(x,t)\in\bar{V}\cap\partial_pD$ we have

$$\bar{\omega}_*(P)=\frac{1}{C_1}\log[C_1\{N\psi_*(P)+\phi(x)\}+1]+C_2|t-t_0|^2+\beta^*(P_0)+\frac{\varepsilon}{2}$$

$$\geqq\beta^*(P_0)+\frac{\varepsilon}{2}>\beta(P),$$

which shows that $\bar{\omega}(P)$ satisfies (ii).

For (iii), let $P=(x,t)\in D\cap\partial V$. Note that if $|x-x_0|\geqq\delta$, then $\phi(x)\geqq(1/C_1)[\exp(CC_1\delta^2/\delta^2)-1]$. Hence V is in the cylinder $|x-x_0|<\delta$. But ∂V may meet the upper and lower boundary of U. If $P\in\partial V$ is on the upper or lower boundary of U, then

$$\bar{\omega}(P)=\frac{1}{C_1}\log[C_1\{N\psi(P)+\phi(x)\}+1]+C_2\eta^2+\beta^*(P_0)+\frac{\varepsilon}{2}.$$

We shall take C_2 so large that $C_2\eta^2+\beta^*(P_0)+\varepsilon/2>L$ (and $-C_2\eta^2+\beta_*(P_0)-\varepsilon/2<K$). If $P\in\partial V$ is not on either the upper or the lower boundary of U, then $N\psi(P)+\phi(x)\geqq(1/C_1)(\exp CC_1-1)$. In this case

$$\varpi(P) \geqq \frac{1}{C_1} \log \left[C_1 \frac{1}{C_1} (\exp CC_1 - 1) + 1 \right] + C_2 |t - t_0|^2 + \beta^*(P_0) + \frac{\varepsilon}{2}$$

$$\geqq C + \beta^*(P_0) + \frac{\varepsilon}{2} = L - K + \beta^*(P_0) + \frac{\varepsilon}{2} \geqq L + \frac{\varepsilon}{2}.$$

Thus we have $\varpi^*(P) > L$ on $D \cap \partial V$, which proves (iii).

We shall now prove (iv). Since

$$\frac{\partial \varpi}{\partial x_i} = \frac{\dfrac{\partial}{\partial x_i}(N\psi + \phi)}{C_1(N\psi + \phi) + 1},$$

$$\frac{\partial^2 \varpi}{\partial x_i \partial x_j} = \frac{\dfrac{\partial^2}{\partial x_i \partial x_j}(N\psi + \phi)}{C_1(N\psi + \phi) + 1} - \frac{C_1 \dfrac{\partial}{\partial x_i}(N\psi + \phi) \dfrac{\partial}{\partial x_j}(N\psi + \phi)}{\{C_1(N\psi + \phi) + 1\}^2},$$

and

$$\frac{\partial \varpi}{\partial t} = \frac{N \dfrac{\partial \psi}{\partial t}}{C_1(N\psi + \phi) + 1} + 2C_2(t - t_0),$$

we have

$$\boldsymbol{P}\,\varpi = \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 \varpi}{\partial x_i \partial x_j} - \frac{\partial \varpi}{\partial t} = \frac{N \left\{ \sum a_{ij}(x, t) \dfrac{\partial^2 \psi}{\partial x_i \partial x_j} - \dfrac{\partial \psi}{\partial t} \right\}}{C_1(N\psi + \phi) + 1}$$

$$+ \frac{\sum a_{ij}(x, t) \dfrac{\partial^2 \phi}{\partial x_i \partial x_j}}{C_1(N\psi + \phi) + 1} - C_1 \frac{\sum a_{ij}(x, t) \dfrac{\partial}{\partial x_i}(N\psi + \phi) \dfrac{\partial}{\partial x_j}(N\psi + \phi)}{\{C_1(N\psi + \phi) + 1\}^2}$$

$$-2C_2(t - t_0).$$

Since

$$\frac{\partial \phi}{\partial x_i} = \frac{2C}{\delta^2} \exp\,(CC_1 |x - x_0|^2 / \delta^2)(x^i - x_0^i)$$

and

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \left\{ 2\delta_{ij} + 4(x^i - x_0^i)(x^j - x_0^j) \frac{CC_1}{\delta^2} \right\} \frac{C}{\delta^2} \exp\,(CC_1 |x - x_0|^2 / \delta^2),$$

we have

$$\sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 2 \sum_{i=1}^{n} a_{ii}(x, t) \frac{C}{\delta^2} \exp\,(CC_1 |x - x_0|^2 / \delta^2)$$

$$+ \frac{4C^2 C_1}{\delta^4} \left[\sum_{i \neq j} a_{ij}(x, t)(x^i - x_0^i)(x^j - x_0^j) \right]$$

$$\cdot \exp\,(CC_1 |x - x_0|^2 / \delta^2).$$

Set $A_1 = \sup \sum a_{ii}(x, t)$, $A_2 = \sup \sum a_{ij}(x, t) \xi_i \xi_j$ for $(x, t) \in D$, $|\xi| = 1$. Then

$$\sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 \phi}{\partial x_i \partial x_j} \leqq \frac{C}{\delta^2} (2A_1 + 4A_2 CC_1 |x - x_0|^2 / \delta^2) \exp\,(CC_1 |x - x_0|^2 / \delta^2).$$

Therefore

$$\boldsymbol{P}\varpi - f\left(x, t, \varpi(x,t), \frac{\partial\varpi}{\partial x_1}(x,t), \cdots, \frac{\partial\varpi}{\partial x_n}(x,t)\right)$$
$$\leqq \frac{1}{C_1(N\psi+\phi)+1}\left[-N+\frac{C}{\delta^2}(2A_1+4A_2CC_1)e^{CC_1}\right]$$
$$+(B-C_1)\frac{\sum a_{ij}(x,t)\frac{\partial}{\partial x_i}(N\psi+\phi)\frac{\partial}{\partial x_j}(N\psi+\phi)}{\{C_1(N\psi+\phi)+1\}^2}+\Gamma-2C_2(t-t_0).$$

Take $C_1 \geqq B$, and note that $C_1(N\psi+\phi)+1 \leqq e^{CC_1}$ in $D\cap V$. If we take N so large that

$$\left\{\frac{C}{\delta^2}(2A_1+4A_2CC_1)+(\Gamma+2C_2\eta)\right\}e^{CC_1}<N,$$

then we have

$$\boldsymbol{P}\varpi - f\left(x, t, \varpi(x,t), \frac{\partial\varpi}{\partial x_1}, \cdots, \frac{\partial\varpi}{\partial x_n}\right)<0 \qquad \text{on } D\cap V.$$

This completes the proof.

Corollary. *If $D\subset R^{n+1}$ is a bounded domain such that each point of its parabolic boundary $\partial_p D$ is parabolically touchable, then to each point of $\partial_p D$ we can construct barrier functions for the equation $\boxdot u = f(x, t, u, (\partial u/\partial x_1), \cdots, (\partial u/\partial x_n))$, where f is assumed to satisfy the same condition stated in the theorem, that is, $|f(x,t,u,p)|\leqq B|p|^2+\Gamma$.*

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