17. Generalized Vector Field and its Local Integration

By Akira ASADA
Department of Mathematics, Sinsyu University, Matumoto

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In this note, we give a generalization of the notion of vector field for a (topological) manifold with a fixed metric and treat the local existence of its integral curve. It also gives a generalization of the notion of tangent of a curve and it allows to consider the tangents at the origin of $\boldsymbol{R}^2$ of the curves such as $r\theta=1$ or the graph of $x\sin(1/x)$. Part of this note has been exposed in [3] and the details of the other part (together with the global studies) will appear in Journal of the Faculty of Science, Shinshu University, vol. 7 under the title "Generalized integral curves of generalized vector fields".

1. d_ρ-smooth functions. We denote by M a connected paracompact n-dimensional topological manifold. By [2] (for the notations, see also [1]), we may choose a metric ρ of M such that by which the topology of M is given and satisfy

(i) *If $\rho(x_1, x_2)\leqq 1$, then there is unique path γ given by $\varphi: \boldsymbol{I}\to M$ such that which join x_1 and x_2 and*

$$\rho(x_1, x_2)=\int_\gamma \rho=\lim_{|t_i-t_{i-1}|\to 0}\sum_{i=1}^m \rho(\varphi(t_i), \varphi(t_{i-1})),$$
$$0=t_0<t_1<\cdots<t_{m-1}<t_m=1.$$

(ii) *To regard ρ to be an Alexander-Spanier 1-cochain of M, if γ is a curve of M such that $\int_\gamma k_a\delta\rho=0$, $a\in\gamma$, then there is a curve γ' which contains γ and*

$$\int_{\gamma'}\rho=\infty, \quad \int_{\gamma'} k_a\delta\rho=0, \quad a\in\gamma'.$$

In $M\times M$, we set $s(M)=\{(x, y)\,|\,\rho(x, y)=1, x\in M\}$. $s(M)$ is the tatal space of an S^{n-1}-bundle over M and its associate $C(S^{n-1})$-bundle is denoted by $C(s(M))$. Here, $C(S^{n-1})$ means the Banach space of continuous functions on S^{n-1} with the compact open topology. Then we can define the Gâteaux-differential d_ρ with respect to ρ (cf. [4], [5]) as the map from the space of functions on M to the space of cross-sections of $C(s(M))$ as follows.

$$d_\rho f(x, y)=\lim_{t\to\infty}\frac{1}{t}\{f(r_{x,y,t})-f(x)\}, \tag{1}$$

where $r_{x,y,t}$ means the point on the curve which joins x and y with the length 1 such that $\rho(x, r_{x,y,t})=t$.

Definition. A function f on M is called d_ρ-smooth or $C(S^{n-1})$-smooth if $d_\rho f$ is a continuous cross-section of $C(s(M))$.

We note that if M is smooth and ρ is the geodesic distance of some Riemannian metric on M, then f is d_ρ-smooth if and only if f is smooth. We denote the space of all $C(S^{n-1})$-smooth functions on M by $C_{C(S^{n-1})}(M)$.

Theorem 1. *$C_{C(S^{n-1})}(M)$ is a dense subring of $C(M)$, the space of all continuous functions on M with the compact open topology.*

2. Generalized vector field. We call a function f on M to be d_ρ-differentiable if $d_\rho f(x)$ exists at every point of M. The space of d_ρ-differentiable functions on M is denoted by $C_\rho(M)$.

Lemma 1. *If $f \in C_\rho(M)$, then the function $\|d_\rho f\|$ given by*

$$\|d_\rho f\|(x) = \max_{y,\rho(x,y)=1} |d_\rho f(x, y)|,$$

is locally bounded.

Definition. A linear operator X from $C_\rho(M)$ into $M_{\text{loc.}}(M)$, the space of locally bounded functions on M, is called a generalized vector field, or a $C(S^{n-1})$-vector field, on M if it satisfies

(i) X is a closed operator.

(ii) $(Xf)(a)$ is equal to 0 if $|f(x)-f(a)|=o(\rho(a, x))$ at a.

(iii) $X(fg)$ is equal to $fX(g)+gX(f)$.

We denote the dual bundle of $C(s(M))$ by $C^*(s(M))$. It is a $C^*(S^{n-1})$-bundle over M.

Theorem 2. *If X is a generalized vector field on M, then there exists a cross-section $\xi(x)$ of $C^*(s(M))$ such that*

$$Xf(x) = \langle \xi(x), d_\rho f(x) \rangle. \tag{2}$$

Conversly, if $\xi(x)$ is a cross-section of $C^*(s(M))$, then to set $Xf(x) = \langle \xi(x), d_\rho f(x) \rangle$, X is a generalized vector field on M.

Definition. If a generalized vector field X is given by $Xf(x) = \langle \xi(x), d_\rho f(x) \rangle$, then we set

$$\xi(x) = \text{rep. } X.$$

3. Generalized tangent. Let γ be a curve of M given by $\varphi: I \to M$, $I=[0, 1]$ and $\varphi(0)=a$, then if the limit

$$\lim_{s\to 0} \frac{1}{s}\left[\lim_{h\to 0}\int_h^s \frac{1}{t}\{f(\varphi(t)) - f(a)\}dt\right]$$

exists for any d_ρ-differentiable function f of M at a, then there exists a positive measure ξ on $S_a=\{y \mid \rho(a, y)=1\}$ such that

$$\langle \xi, d_\rho f(a) \rangle = \lim_{s\to 0} \frac{1}{s}\left[\lim_{h\to 0}\int_h^s \frac{1}{t}\{f(\varphi(t)) - f(a)\}dt\right]. \tag{3}$$

Definition. We call the above ξ to be the generalized tangent of γ at a.

Example 1. If γ is smooth at a, then the generalized tangent of γ at a is $c\delta_y$, where c is a constant and δ_y is the Dirac measure on S_a with the carrier $\{y\}$.

Example 2. If γ is given by $r\theta=1$ in $\boldsymbol{R}^2$, then the generalized tangent of γ at 0, the origin of $\boldsymbol{R}^2$, is $(1/2\pi)d\theta$.

Example 3. If γ is the graph of $x\sin(1/x), x>0$, then the generalized tangent of γ at 0 is the measure on S^1 with the carrier $-\pi/4 \leqq\theta\leqq\pi/4$ and given there by $(1/\pi\cos^2\theta\sqrt{\cos(2\theta)})d\theta$.

Note. Prof. Uchiyama teaches the author that if $xf(1/x)$ is almost periodic in the sense of Besicovič, then the graph of $f(x)$ has the generalized tangent at the origin. On the other hand, it is also shown that if f is Lipschitz continuous near the origin, then the graph of f also has the generalized tangent at the origin.

Theorem 3. *If ξ is a positive measure on S_a, then there exists a curve on M such that its generalized tangent at a is ξ.*

4. Local integration of the generalized vector field. We assume $M=\boldsymbol{R}^n$ and ρ is the euclidean metric. Hence we have

$$s(\boldsymbol{R}^n)=\boldsymbol{R}^n\times S^{n-1}.$$

In $C(S^{n-1})$, we denote the subspace consisted by the linear functions by $l(S^{n-1})$ and decompose $C^*(S^{n-1})$ as follows: *To define a subspace $l^*(S^{n-1})$ of $C^*(S^{n-1})$ by*

$$l^*(S^{n-1})=\left\{\sum_{i=1}^{n} c_i\delta_i \mid c_i\in\boldsymbol{R}\right\},$$

where δ_i is the Dirac measure of S^{n-1} with the carrier at $(0,\cdots,0,\overset{i}{1},0,\cdots,0)$, and set

$$C^*(S^{n-1})=l^*(S^{n-1})\oplus l(S^{n-1})^{\perp}. \tag{4}$$

In (4), we denote the projections from $C^*(S^{n-1})$ to $l^*(S^{n-1})$ and $l(S^{n-1})^{\perp}$ by p_1 and p_2. Then, for a generalized vector field X, *rep.* $X=\xi(x)$, on $\boldsymbol{R}^n$, we define the generalized vector fields $D(X)$ and $S(X)$ by

$$(D(X)f)(x)=\langle p_1(\xi(x)), d_\rho f(x)\rangle,$$
$$(S(X)f)(x)=\langle p_2(\xi(x)), d_\rho f(x)\rangle.$$

Then we have

Theorem 4. *We may consider X to be a usual vector field on $\boldsymbol{R}^n$ if and only if $X=D(X)$. On the other hand, if $X=S(X)$ and f is d_ρ-differentiable on $\boldsymbol{R}^n$ then Xf is equal to 0 almost everywhere on $\boldsymbol{R}^n$.*

On the other hand, since $l^*(S^{n-1})=\boldsymbol{R}^n$, we consider $\boldsymbol{R}^n$ to be a subspace of $C^*(S^{n-1})$ by (4). Then we can extend $\xi(x)$ (=rep. X) to be a function $\xi^{\#}(x)\colon C^*(S^{n-1})\to C^*(S^{n-1})$ and if the function $\|\xi\|(x)$ satisfies the Lipschitz condition, then the equation

$$\frac{du(t)}{dt}=\xi^{\#}(u(t)) \tag{5}$$

has unique solution in $C^*(S^{n-1})$ under the given initial condition.

Definition. We call the solution of (5) with the initial condition $u(0)=a$ to be the integral curve of X starts from a.

Then we obtain

Theorem 5. *If $X=D(X)$, then the generalized integral curve of X is the usual integral curve of X. On the other hand, if $X=S(X)$ and $u(t)$ is a solution of* (5), *then we get $p_1(u(t))=p_1(u(0))$ for all t.*

References

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