16. *Cohomology of Lie Algebras over a Manifold*

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In 1970 M. V. Losik [3] has generalized the de Rham complex in the higher order jet spaces and determined its cohomology group completely. Immediately later, Gelfand and Fuks [2] have given an alternative proof to Losik's result from the viewpoint of their general theory concerning the cohomology of vector fields. Actually, they have not only reformulated and extended Losik's result in terms of differential forms, but also established an interesting relation between a representation of the general linear group and the one of the vector fields induced from it; the latter representation depends essentially on the first jet of the tangent bundle. In view of the cohomology theory of Lie algebras, from these representations canonically arise two complexes: one is associated to the Lie algebra of formal vector fields without constant terms and the other is the Lie algebra of vector fields. Gelfand and Fuks have further clarified in the cited paper that the cohomology groups of these two complexes stand in close relation connected by some spectral sequence.

We shall here generalize their results to the case where the representations are concerned with the higher ordre jet of the tangent bundle. Moreover, we shall formulate and prove the finite dimensionality of the cohomology groups of vector fields associated to these representations in a considerably general form.

1. Let $\mathfrak{a}_n$ be the Lie algebra of formal vector fields with n indeterminates. That is, $\mathfrak{a}_n$ consists of the elements with the form $\xi = \sum_{\mu=1}^{n} \xi^{\mu}(x_1, \cdots, x_n)\partial/\partial x_{\mu}$, $\xi^{\mu} \in \boldsymbol{R}[[x_1, \cdots, x_n]]$, and has the bracket rule induced from the usual differentiation. We consider $\mathfrak{a}_n$ as a topological Lie algebra where $\mathfrak{a}_n$ is endowed with the Krull topology. Let $\mathfrak{m}$ be the maximal ideal of $\boldsymbol{R}[[x_1, \cdots, x_n]]$ and set $L_k = \mathfrak{m}^{k+1}\mathfrak{a}_n$. Then $\mathfrak{a}_n$ is a simple Lie algebra and each L_k $(k=0, 1, 2, \cdots)$ becomes an ideal of L_0. Moreover, we have $L_0/L_1 \cong \mathfrak{gl}(n; \boldsymbol{R})$, which is in turn obtained from a splitting $L_0 = \mathfrak{gl}(n; \boldsymbol{R}) \oplus L_1$. Let V be a finite-dimensional vector space over $\boldsymbol{R}$. Let $C^p(L_0, V)$ be the space consisting of the continuous alternating p-linear maps from L_0 to V. Then we have

$$C^p(L_0, V) = \varinjlim C_k^p(L_0, V),$$

where $C_k^p(L_0, V)$ denotes the subspace of $C^p(L_0, V)$, the elements of which are characterized as the liftings of alternating p-linear maps from L_0/L_k to V.

It should be noted that any continuous Lie algebra representation φ of L_0 on V is obtained as the lifting of a representation of L_0/L_h on V for some h. We say that a representation φ of L_0 on V is *completely reducible* if $\varphi|\mathfrak{gl}(n;\boldsymbol{R})$ is completely reducible. For any integers $f_1 \geqq \cdots \geqq f_m > 0$, denote by $[f_1, \cdots, f_m]$ the irreducible representation of $\mathfrak{gl}(n;\boldsymbol{R})$ corresponding to the Young diagram $f_1 \geqq \cdots \geqq f_m > 0$. Also we denote by δ Trace $(\delta \in \boldsymbol{R})$ the one-dimensional representation of $\mathfrak{gl}(n;\boldsymbol{R})$ assigning δ to Trace A $A(\in \mathfrak{gl}(n;\boldsymbol{R}))$. Set

$$[f_1, \cdots, f_m; \delta] = [f_1, \cdots, f_m] \otimes 1 + 1 \otimes \delta \text{ Trace}.$$

If a given representation φ of L_0 on V is completely reducible, then we have a direct decomposition

$$\varphi|\mathfrak{gl}(n;\boldsymbol{R}) = \sum \oplus [f_1, \cdots, f_m; \delta]$$

in irreducible components. Among $[f_1, \cdots, f_m; \delta]$ occurring in the right side, we observe the irreducible representations $[f_1, \cdots, f_m; \delta]$ satisfying

i) $f_1 + \cdots + f_m + n\delta \geqq 0$

ii) $n\delta$ is an integer,

and denote by Δ the subfamily formed by those $[f_1, \cdots, f_m; \delta]$.

Associated to the representation φ, we can construct a complex $\{C^p(L_0, V), d\}$ by virtue of the cohomology theory of Lie algebras. $H^*(L_0, V) = \sum \oplus H^p(L_0, V)$ denotes the cohomology group of the complex $\{C^p(L_0, V), d\}$. It is verified that if φ is the lifting of some representation of L_0/L_h on V, then for $k \geqq h$ $C^p_k(L_0, V) \xrightarrow{d} C^{p+1}_k(L_0, V)$ $(p=0, 1, 2, \cdots)$ is well-defined, so that $\{C^p_k(L_0, V), d\}$ becomes a subcomplex.

Theorem 1. *Let φ be completely reducible and assume that φ is the lifting of a representation of L_0/L_h on V. Then the following statements are valid:*

i) *Put*

$$k_0 = \text{Max}\left\{h, \underset{[f_1, \cdots, f_m; \delta] \in \Delta}{\text{Max}} n\left(\sum_{i=1}^m f_i + n\delta + 1\right)\right\}.$$

Then for $k \geqq k_0$ the inclusion of the subcomplex $\{C^p_k(L_0, V), d\}$ in $\{C^p(L_0, V), d\}$ induces the isomorphism on the cohomological level.

ii) *If $\Delta = \phi$, then $H^*(L_0, V) = 0$.*

The situation i) is often expressed that $\{C^p(L_0, V), d\}$ has the stable jet range $k \geqq k_0$.

Corollary 1. $\dim H^*(L_0, V) < +\infty$.

Corollary 2. *If all $n\delta$ are not integers, we have $H^*(L_0, V) = 0$.*

Corollary 3. $\sum (-1)^p \dim H^p(L_0, V) = 0$.

2. Let $\text{Diff}_n(0)$ be the group germ of local diffeomorphisms of $\boldsymbol{R}^n$ around 0. For $\varphi \in \text{Diff}_n(0)$, let $[\varphi]_h$ be the h-jet of φ at 0. The totality of $[\varphi]_h$ forms a Lie group $G(h)$ under the composition rule $[\varphi]_h\circ[\psi]_h=[\varphi\circ\psi]_h$. $G(h)$ is really obtained as successive extensions of $GL(n;\boldsymbol{R})$ by vector groups. The Lie algebra of $G(h)$ is canonically identified with L_0/L_h. Any element of $\mathrm{Diff}_n(0)$ naturally operates on a germ of vector fields of $\boldsymbol{R}^n$ around 0.

Let M be a smooth manifold with countable basis. Let $\tau(M)$ be the tangent bundle of M and $J^k(\tau(M))$ the k-th jet bundle of $\tau(M)$. Note that the structural group of $J^{h-1}(\tau(M))$ is reducible to $G(h)$. Let $P(h)$ be the principal $G(h)$-bundle associated to $J^{h-1}(\tau(M))$.

Now assume that a finite-dimensional representation ρ of $G(h)$ on V be given. Then we have the Lie algebra representation $d\rho$ of L_0/L_h on V. (The lifting of $d\rho$ to L_0 is also denoted by the same notation $d\rho$.) We have thus obtained the complex $\{C^p(L_0, V), d\}$ associated to the representation $d\rho$.

On the other hand, define the vector bundle W over M by

$$W=P(h)\times_\rho V.$$

Take a fixed base $\{e_1,\cdots,e_s\}$ of V. On any local coordinates neighborhood $\{U;x_1,\cdots,x_n\}$ with $\sum|x_i|^2<1$, the canonical local triviality is induced on $P(h)$, hence on W. Let $\{\tilde{e}_1,\cdots,\tilde{e}_s\}$ be the local basis of W on U. Assume that $d\rho$ is explicitly given by

$$d\rho(\{\xi_A^\mu\})=\sum C_{\beta\mu}^{\alpha A}\xi_A^\mu e^\beta\otimes e_\alpha,$$

where A runs over multi-indices, $\mu=1,\cdots,n$ and e^β denotes the dual base of e_α. Let $\mathfrak{A}(M)$ be the Lie algebra of vector fields on M. For $\xi\in\mathfrak{A}(M)$ and $\sigma\in\Gamma(W)$, we put on U

$$\rho^\#(\xi)\sigma=\sum \xi^\mu\frac{\partial\sigma^\alpha}{\partial x_\mu}\tilde{e}_\alpha+\sum C_{\beta\mu}^{\alpha A}\frac{\partial^{|A|}\xi^\mu}{\partial x^A}\sigma^\beta\tilde{e}_\alpha.$$

Proposition 1. *$\rho^\#$ gives rise to a well-defined representation of $\mathfrak{A}(M)$ on $\Gamma(W)$ in the sense of* [4].

Hence, according to the terminology in [4], we can obtain a differential complex $\{C^p[\tau(M), W], d\}$ associated to the differential representation $\rho^\#$. We denote its cohomology group by $H^*(\tau(M), W)=\sum\oplus H^p(\tau(M), W)$.

3. Therefore, given a representation ρ of $G(h)$ on V, we have canonically two complexes $\{C^p(L_0, V), d\}$ and $\{C^p[\tau(M), W], d\}$; the former is of local feature while the latter is of global one. These two complexes, however, are closely related.

Theorem 2. *Assume that $\rho|GL(n;\boldsymbol{R})$ is completely reducible. Then we have the following assertions:*

i) *If M is compact, then* $\dim H^*(\tau(M), W)<+\infty$. *More precisely, if M is compact and simply-connected, then*

$$\dim H^p(\tau(M), W)\leqq\sum_{q+r=p}\dim H^q(M;\boldsymbol{R})\times\dim H^r(L_0, V).$$

ii) *If* $\{C^p(L_0, V), d\}$ *has the stable jet range* $k \geqq k_0$, *then* $\{C^p(\tau(M), W), d\}$ *has also the stable jet range* $k \geqq k_0$ (cf. [4]).

iii) *In the case where* Δ *becomes empty for the complex* $\{C^p(L_0, V), d\}$, *then* $H^*(\tau(M), W)=0$.

iv) *If* M *is compact and simply connected, then*

$$\sum(-1)^p \dim H^p(\tau(M), W)=0.$$

Corollary. *If* φ *is obtained by the lifting of a non-trivial contravariant representation of* $GL(n; \boldsymbol{R})$, *then we have* $H^*(\tau(M), W)=0$.

4. Similar results hold in some other cases. There are various problems related to the above results which suggest the further development in those directions. We shall give the proofs to the above results and carry on the study on those subjects in a series of papers now under preparation.

References

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