# 19. *On the Family of the Solution-Curves of the Integral Inequality*

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(Comm. by K. KUNUGI, M.J.A., Feb. 12, 1954)

A certain generalization of the theorem of Kneser on the differential inequality was shown by Prof. M. Hukuhara.[1] In this note, we shall generalize it to the case of integral inequality

$$(1)\qquad |u(x)-f(x)-\int_0^x K(x,t,u(t))dt|\leq p(x)$$

where the functions $f$, $u$ and $K$ represent $n$-dimensional vectors, while $x$, $t$ and $p$ are real; $f(x)$ is continuous in $0\leq x\leq 1$, $K(x,t,u)$ is bounded and continuous in the domain $D$:

$$0\leq t\leq x\leq 1,\qquad |u|<\infty,$$

$p(x)$ is continuous in the interval $0\leq x\leq 1$.

*Suppose that the family* $\mathfrak{F}$ *of* $f(x)$ *is a compact continuum in* (C) *and* $\mathfrak{U}$ *is the family of the totality of the solution-curves*[2] *of* (1) *with* $f(x)\in\mathfrak{F}$. *Then,* $\mathfrak{U}$ *is also a compact continuum in* (C).

*cf.* (C) denotes the space of continuous functions on $0\leq x\leq 1$ with the norm $\|f\|=\max_{0\leq x\leq 1}|f(x)|$.

It is evident that the family $\mathfrak{U}$ is a closed and compact set in (C). If $\mathfrak{U}$ is not a continuum, $\mathfrak{U}$ must be the sum of two closed, disjoint and non void sets $\mathfrak{U}_1$ and $\mathfrak{U}_2$. Let $\mathfrak{F}_i$ be the family of the functions $f_i(x)$ whose corresponding solutions are in $\mathfrak{U}_i (i=1, 2)$. Then $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are closed and $\mathfrak{F}=\mathfrak{F}_1\cup\mathfrak{F}_2$. As $\mathfrak{F}$ is a continuum, there exists $f_0$ such that

$$f_0\in\mathfrak{F}_1\cap\mathfrak{F}_2.$$

The family $\mathfrak{U}_0$ of the solution-curves corresponding to $f_0$ contains an element of $\mathfrak{U}_1$ and an element of $\mathfrak{U}_2$. Therefore, if we can prove that $\mathfrak{U}_0$ is a continuum, $\mathfrak{U}_0$ must contain an element which does not belong to $\mathfrak{U}$. This contradicts to $\mathfrak{U}_0\subseteqq\mathfrak{U}$. Therefore, it is sufficient to prove that $\mathfrak{U}_0$ is a continuum, i.e. the solution-curves $\mathfrak{U}_0$ of the following integral inequality

$$(2)\qquad |u(x)-f(x)-\int_0^x K(x,t,u(t))dt|\leq p(x)$$

1) M. Hukuhara: Sur une généralisation d'un théorème de Kneser, Proc. Japan Acad., **29**, 154 (1953).

2) 3) For the existence of such solutions, see T. Satô's "Sur les équations integrales non-linéaires de Volterra" (forthcoming in «Compositio Mathematica»).

is a continuum.

As $\mathfrak{U}_0$ is clearly a closed set in (C), if $\mathfrak{U}_0$ is not a continuum, $\mathfrak{U}_0$ must be sum of two closed and disjoint sets $\mathfrak{U}_0^1$ and $\mathfrak{U}_0^2$. Take $u_1(x)$ and $u_2(x)$ in $\mathfrak{U}_0^1$ and $\mathfrak{U}_0^2$ respectively. And set $\max_{0\leq x\leq 1}|f(x)|=F$, $|K(x, t, u)|\leq M$ and $\Omega$: $0\leq t\leq x\leq 1$, $u\leq F+M$.

Consider the integral equation

$$u(x)=f(x)+\int_0^{\alpha} K(x, t, u(t))dt+\int_{\alpha}^{x} K_n(x, t, u(t))dt \qquad (i=1, 2) \tag{3}$$

where $0\leq\alpha\leq 1$, $K_n(x, t, u)$ satisfies the Lipschitz's condition with respect to $u$ and converges to $K(x, t, u)$ uniformly in $\Omega$.

Put[3)]

$$g_n^i(x, \alpha)=\begin{cases} u_i(x) & \text{for } 0\leq x\leq \alpha \\ \text{solution of (3)} & \text{for } \alpha\leq x\leq 1, \end{cases}$$

then

$$g_n^i(x, 1)=u_i(x) \qquad (i=1, 2)$$
$$g_n^1(x, 0)=g_n^2(x, 0).$$

Because $g_n^i(x, \alpha)$, considered as a function of $x$, is continuous in (C) with respect to $\alpha$, the sets

$$\mathfrak{G}_n=\{g_n^i(x, \alpha);\quad i=1, 2\}$$

is a continuum which contains $u_1(x)$ and $u_2(x)$.
Take two open sets $\mathfrak{H}_1$ and $\mathfrak{H}_2$ in (C) such

$$\mathfrak{H}_1\supset\mathfrak{U}_0^1,\quad \mathfrak{H}_2\supset\mathfrak{U}_0^2,\quad \mathfrak{H}_1\frown\mathfrak{H}_2=O.$$

Then there exists an element $g_n(x, \alpha_n)$ in $\mathfrak{G}_n$ which is not contained in $\mathfrak{H}_1\smile\mathfrak{H}_2$. The family $\{g_n(x, \alpha_n)\}$ is, as easily be seen, equi-bounded and equi-continuous, so that we can take a uniformly convergent sequence whose limit $g(x)$ is not contained in $\mathfrak{U}_0$, while $g(x)$ is a solution of (2) from its construction. This is a contradiction.

q.e.d.

From this theorem we can easily have the following corollary.

*Let $C_0$ be a solution-curve of* (1). *If there are more than two solutions, there exists, for any small positive number $\varepsilon$, a solution-curve $C$ such that $0<\rho(C, C_0)<\varepsilon$, where $\rho(C, C_0)$ is the distance of $C$ and $C_0$ in the space* (C).

We wish to express our gratitude to Prof. T. Satô for his kind guidance.