18. *Conjugate Spaces of Operator Algebras*

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By the representation theorem for (AL)-space,[4] the conjugate space $R(\Omega)$ of the Banach space $C(\Omega)$ composed of all continuous complex valued functions vanishing at infinity on a locally compact Hausdorff space Ω i.e. the space of all bounded complex Radon measures on Ω is isomorphic to $L^1(\Gamma)$ on a suitable localizable measure space Γ. Hence the conjugate space of $R(\Omega)$ is isomorphic to $L^\infty(\Gamma)$. As the measure space Γ is localizable, $L^\infty(\Gamma)$ considered as the set of multiplication operators on $L^2(\Gamma)$ is a maximal abelian subring in the ring of all bounded operators on $L^2(\Gamma)$,[8] this implies $L^\infty(\Gamma)$ is a weakly closed operator algebra i.e. a W^*-algebra. On the other hand, the double conjugate space of the Banach space composed of all completely continuous operators on a Hilbert space H is isomorphic to the space $B(H)$ of all bounded operators on H.[2][7] From these special cases, we get naturally the following conjecture: *Is the double conjugate space of a uniformly closed self-adjoint operator algebra or equivalently a B^*-algebra always isomorphic to a W^*-algebra considered as a Banach space?* The affirmative of this conjecture was announced by S. Sherman,[9] but its detailed proof is not published yet now. In § 2 of this note, we give a proof of this theorem. By a letter from S. Sherman the author learned that his original proof is essentially same with our own. In this occasion, we want to express our hearty thanks for his kind regards. Recently J. Dixmier has shown that a W^*-algebra considered as a Banach space is always isomorphic to the conjugate space of all ultra-weakly continuous linear functionals.[3] Observing this remarkable property of W^*-algebras, we give in § 3 a characterization of W^*-algebras. Even though this characterization does not depend on the algebraic structure of the algebra,[5] it seems for us to have some interestings especially from the view point of the non-commutative integration theory. The detailed explanations of this point will be shown in the forthcoming paper.

1. In this section, we give a theorem due to J. Dixmier as a preparation for the followings. Let $A, \bar{A}$ be a Banach space and its conjugate space respectively and by $\sum, \bar{\sum}$ denote the unit sphere of each space. For a closed subspace V of $\bar{A}$ we define its *characteristic* r by

$$r = \inf_{x \in \Sigma} \sup_{f \in V \cap \Sigma} \frac{|f(x)|}{||x||}$$

Then r satisfies always $0 \leq r \leq 1$. We call a weakly dense subspace V of $\bar{A}$ *minimal* if any other subspaces of V is not weakly dense in $\bar{A}$. About many interesting properties of these concepts we may refer to Dixmier's paper.[1)]

We need the Dixmier theorem in the following form.[1)]

Theorem. *The necessary and sufficient condition for a Banach space A to be isomorphic to a conjugate space of a suitable Banach space is to exist in $\bar{A}$ a minimal subspace V with its characteristic $r=1$. In this case, A is isomorphic to the conjugate space of V.*

Then, as a corollary of this theorem we get: *The unit sphere $\sum$ of A is compact by the topology $\sigma(A, V)$ for V stated in the theorem.*

2. Analogously to the Jordan decomposition of a measure, we get the following decomposition of a linear functional on a B^*-algebra.

Lemma. *Let A be a B^*-algebra. Then every bounded linear functional on A can be expressed by a finite linear combination of states of A.*

Proof. The state space S of A is generally a locally compact Hausdorff space. Let $\sigma(a)$ be the value of $a \in A$ for a state σ. Defining $a(\sigma)=\sigma(a)$ for each point σ of the state space, we can associate for every $a \in A$ a continuous function $a(\sigma)$ on Ω vanishing at infinity. Then clearly we get the followings:

(1) $(\alpha a+\beta b)(\sigma)=\alpha a(\sigma)+\beta b(\sigma)$ for $a, b \in A$ and complex numbers α, β.

(2) $a(\sigma)$ is real valued for a hermitian element a.

(3) $a(\sigma)$ is non-negative for a hermitian non-negative element a.

Generally $|a(\sigma)| \leqq ||a||$. Let us suppose A be a uniformly closed operator algebra on a certain Hilbert space H and $a \in A$ be a hermitian operator, then, as well known,

$$||a|| = \sup_{\varphi \in H, ||\varphi||=1} |<a\varphi, \varphi>|,$$

where $<\ ,\ >$ shows the inner product of H. Hence the real Banach space A^r composed of all hermitian elements of A is isometrically isomorphic to a subspace M of the space $C^r(S)$ of all real valued continuous functions vanishing at infinity on S and the positiveness is preserved by this isomorphism.

Let f be a bounded linear functional on A, then $f=g+ih$ where g, h are real valued linear functionals on A. These g and h can be considered as bounded linear functionals on M and can be extended over $C^r(S)$. Let $\tilde{g}, \tilde{h}$ be these extensions, then $\tilde{g}, \tilde{h}$ are nothing but real Radon measures on S. We decompose $\tilde{g}$ into two positive Radon measures $\tilde{g}_1, \tilde{g}_2$ such as $\tilde{g}=\tilde{g}_1-\tilde{g}_2$ and s_1, s_2 be the normalization of $\tilde{g}_1, \tilde{g}_2$ respectively (i.e. $\tilde{g}_i=\alpha_i s_i$, $\alpha_i>0$, $\|s_i\|=1$). Hence $\tilde{g}$ is expressed by $\tilde{g}=\alpha_1 s_1-\alpha_2 s_2$ and the restriction of s_i on M gives a state σ_i on A. Similarly we can decompose $\tilde{h}$ into $\tilde{h}=\alpha_3 s_3-\alpha_4 s_4$, hence f is decomposed into $f=\alpha_1\sigma_1-\alpha_2\sigma_2+i\alpha_3\sigma_3-i\alpha_4\sigma_4$ as desired. q.e.d.

For each $\sigma \in S$, we construct a Hilbert space H_σ by the usual method and $a_\sigma^{\#}$ be the standard representation of $a \in A$ on H_σ. $H=\sum_{\sigma \in S} H_\sigma$ be the direct sum of $H_\sigma (\sigma \in S)$ and $a^{\#}$ be the operator on H which plays as same as $a_\sigma^{\#}$ on each component space H_σ. Then the mapping $a \to a^{\#}$ gives a one-to-one representation of A on H and so the norm is preserved by this representation. Let $A^{\#}$ be the image of this representation, then, by the above, A and $A^{\#}$ are isomorphic as Banach spaces especially. Put the weak closure of this operator algebra $A^{\#}$ be W, then we get S. Sherman's theorem.

Theorem I. *The double conjugate* $\overline{\overline{A}}$ *of the Banach space A is isomorphic to the Banach space W.*

Proof. A state σ_0 on A can be expressed $\sigma_0(a)=<a_{\sigma_0}^{\#}\varphi_{\sigma_0}, \varphi_{\sigma_0}>$ by an element φ_{σ_0} in H_{σ_0}. Hence we define $\varphi \in H \equiv \sum_{\sigma \in S} H_\sigma$ as follows: σ_0-component of $\varphi=\varphi_{\sigma_0}$, σ-component of $\varphi=0$ for $\sigma \neq \sigma_0$, then the state σ_0 can be represented as $\sigma_0(a)=<a^{\#}\varphi, \varphi>$ by the element φ of H. A state defined by such an element $\varphi \in H$ is called a *canonical state*. Now, by the above lemma, a linear functional on A can be considered as a finite linear combination of canonical states of $A^{\#}$. Conversely a finite linear combination of canonical states of $A^{\#}$ defines a linear functional on A. In the followings, we identify all finite linear combinations of canonical states of $A^{\#}$ which define an identical linear functional on A, then the conjugate space $\overline{A}$ of the Banach space A is isomorphic to the space V constructed by all classes of finite linear combinations of canonical states of $A^{\#}$.

Let $a_\alpha^{\#} \in A^{\#}$ converges to $w \in W$ weakly, then $<a_\alpha^{\#}\varphi, \eta>$ converges to $<w\varphi, \eta>$ for each $\varphi, \eta \in H$, and so $\sum_{i=1}^{n} <a_\alpha^{\#}\varphi_i, \eta_i>$ converges to $\sum_{i=1}^{n} <w\varphi_i, \eta_i>$ for every $\varphi_i, \eta_i \in H$. Hence all finite linear combinations of canonical states which are identified on $A^{\#}$ define a unique bounded linear functional f on W. In this case, the norm of the linear functional f on $A^{\#}$ is given by $\|f\|_{\#}=\sup_{\|a^{\#}\|\leq 1} |f(a^{\#})|$ and the norm of f on W is $\|f\|_W=\sup_{\|w\|\leq 1} |f(w)|$. On the other hand, the unit sphere of $A^{\#}$ is weakly dense in the unit sphere of W.[6] Hence by the weak continuity of f, we get $\|f\|_{\#}=\|f\|_W$.

As the last step of the proof, we show that the Banach space

W is isomorphic to the conjugate space of V. This is clear from a theorem of Dixmier[3] but we give here a short proof. As every functional $f \in V$ permits a representation $f(a^{\#}) = <a^{\#}\varphi, \eta>$ by $\varphi, \eta \in H$, an element F of the conjugate space $\overline{V}$ of V defines a bounded bilinear functional on H and so defines an operator t on H such as $F(f) = <t\varphi, \eta>$ for every $f \in V$. Moreover, if $g \in V$ is given by $g(a^{\#}) = <a^{\#}a'\varphi, \eta>$ where a' is an operator in the commutor $A^{\#'}$, $F(g) = <ta'\varphi, \eta>$. As $<a^{\#}a'\varphi, \eta> = <a^{\#}\varphi, a'^{*}\eta>$, $F(g) = <t\varphi, a'^{*}\eta> = <a't\varphi, \eta>$. Hence $t \in W$ and the remainder of the proof is evident. q.e.d.

3. In the first place, we introduce a concept which plays a prominent rôle in this section.

Definition. If a closed subspace V of the conjugate space $\overline{A}$ of a B^*-algebra A satisfies the following four conditions:

(i) V is minimal.

(ii) The characteristic r of $V=1$.

(iii) If $f \in V$, then $f = \sum_{i=1}^{n} \alpha_i \sigma_i$ where σ_i are states contained in V and α_i are complex numbers.

(iv) Let $f \in V$ and $a, b \in A$, then $g(x)$ defined by $g(x) = f(a^*xb)$ is contained in V.

Then we call V a *rudimentary subspace* of $\overline{A}$.

Theorem 2. *In order that a B^*-algebra A has a representation as a weakly closed operator algebra, it is necessary and sufficient to have a rudimentary subspace in the conjugate space $\overline{A}$.*

Proof. Necessity. When A is represented as a weakly closed operator algebra $A^{\#}$ on a Hilbert space H, denote the representation of $a \in A$ by $a^{\#}$. Then the Banach space A is isomorphic to the conjugate space of V composed of all ultra-weakly continuous linear functional f of A, which can be described in the following form:

$f(a) = \sum_{i=1}^{\infty} <a^{\#}\varphi_i, \eta_i>$, where $\varphi_i, \eta_i \in H$ satisfy $\sum_{i=1}^{\infty} \| \varphi_i \|^2 < +\infty$, $\sum_{i=1}^{\infty} \| \eta_i \|^2 < +\infty$.[3] Therefore V is minimal in $\overline{A}$ and its characteristic $r=1$.

By the identity

$$4<a\varphi, \eta> = <a(\varphi+\eta), \varphi+\eta> - <a(\varphi-\eta), \varphi-\eta> + i<a(\varphi+i\eta), \varphi+i\eta> - i<a(\varphi-i\eta), \varphi-i\eta>,$$

clearly V satisfies the condition (iii).

If $g(x)$ is a functional defined by $g(x)=f(a^*xb)$ for an ultra-weakly continuous linear functional $f(x) = \sum_{i=1}^{\infty} <a^{\#}\varphi_i, \eta_i>$,

$$g(x) = f(a^*xb) = \sum_{i=1}^{\infty} <a^{\#*}x^{\#}b^{\#}\varphi_i, \eta_i> = \sum_{i=1}^{\infty} <x^{\#}b^{\#}\varphi_i, a^{\#}\eta_i>$$

and

$\sum_{i=1}^{\infty}\|b^{\#}\varphi_i\|^2 \leqq \|b^{\#}\|^2\sum_{i=1}^{\infty}\|\varphi_i\|^2 < +\infty$, $\sum_{i=1}^{\infty}\|a^{\#}\eta_i\|^2 \leqq \|a^{\#}\|^2\sum_{i=1}^{\infty}\|\eta_i\|^2 < +\infty$, V satisfies the condition (iv), too.

Therefore the space V defined as above is a rudimentary subspace in the space $\bar{A}$.

Sufficiency. S be the set of all states contained in a rudimentary subspace V. Then for each state $\sigma \in S$, we construct a Hilbert space H_σ as usual and $H=\sum_{\sigma\in S} H_\sigma$ be the direct sum of these spaces. $a^{\#}$ be the representation of $a \in A$ as an operator on H, then we get an isomorphic representation of A as a uniformly closed operator algebra $A^{\#}$ on H. For, since V is dense in $\bar{A}$, for each non-zero $a \in A$, there exists an $f \in V$ such as $f(a) \neq 0$, hence exists a $\sigma \in S$ such as $\sigma(a) \neq 0$. This concludes $a^{\#} \neq 0$.

Next we show $A^{\#}$ is weakly closed. By the condition (iii) $f \in V$ is represented by a finite linear combination of canonical states on $A^{\#}$, hence the weak operator topology on $A^{\#}$ induced by H is stronger than the topology $\sigma(A, V)$. On the other hand, let a_σ^0 be the standard mapping of $a \in A$ into H_σ and φ_σ be the H_σ-component of $\varphi \in H$, then for each $\varphi \in H$ there exists a $\varphi_n \in H$ which satisfies the following two conditions:

(α) $\|\varphi-\varphi_n\| < \frac{1}{n}$.

(β) H_σ-component of φ_n is all 0 except σ of finite numbers and for non-zero $\varphi_{n\sigma}$ there exists $b \in A$ such as $\varphi_{n\sigma}=b_{n\sigma}^0$.

Similarly for $\eta \in H$ we take out η_n such as $\eta_{n\sigma}=c_{n\theta}^{\sigma}$, $c \in A$ for each non-zero H_σ-component $\eta_{n\sigma}$. Then for $f(a)$ and $f_n(a)$ defined by $<a^{\#}\varphi, \eta>$, $<a^{\#}\varphi_n, \eta_n>$ respectively,

$$f(a) = <a^{\#}\varphi, \eta> = \sum_{\sigma\in S} <a_\sigma^{\#}\varphi_\sigma, \eta_\sigma>$$

$$f_n(a) = <a^{\#}\varphi_n, \eta_n> = \sum_{\sigma\in S} <a_\sigma^{\#}\varphi_{n\sigma}, \eta_{n\sigma}>$$

$$= \sum_{\sigma\in S} \sigma(c^*ab).$$

The latter summation $\sum$ is a finite sum in practice and so $\sum_{\sigma\in S}\sigma(c^*ab) \in V$. Moreover

$$\|f-f_n\| = \sup_{a^{\#}\in A^{\#},\ \|a^{\#}\|\leq 1} |<a^{\#}\varphi, \eta> - <a^{\#}\varphi_n, \eta_n>|$$

$$= \sup_{a^{\#}\in A^{\#},\ \|a^{\#}\|\leq 1} |<a^{\#}\varphi, \eta> - <a^{\#}\varphi, \eta_n> + <a^{\#}\varphi, \eta_n> + <a^{\#}\varphi_n, \eta_n>|$$

$$\leqq \|\varphi\|\frac{1}{n} + \left(\|\eta\| + \frac{1}{n}\right)\frac{1}{n}.$$

This shows V is strongly dense in the set U of all finite linear combination of canonical states. Therefore, on the unit sphere $\sum$ of A, the topology $\sigma(A, V)$ coincides with the weak operator topology $\sigma(A, U)$. $\sum$ is compact by $\sigma(A, V)$, hence so by $\sigma(A, U)$.

Let M be the weak closure of $A^{\#}$ on H, then for each $m \in M$ there exists a directed family $a_\alpha^{\#} \in A^{\#}$ such as $a_\alpha^{\#}$ converges to m weakly and $\|a_\alpha^{\#}\| \leqq \|m\|$ for all α. From the compactness of $\sum$ by $\sigma(A, U)$, a subfamily of $a_\alpha^{\#}$ has a limit point in $A^{\#}$. Therefore $m \in A^{\#}$. This shows the weak closedness of $A^{\#}$. q.e.d.

By Dixmier[3] a state on a W^*-algebra is ultra-weakly continuous if and only if it is normal and the normality of a state is a purely algebraical property. From this fact combined with Theorem 2 it follows:

Theorem 3. *If there exists a rudimentary subspace in the conjugate space of a B^*-algebra, it is unique and is nothing but the set of all finite linear combinations of normal states.*

Remark. It is known that for any commutative W^*-algebra, every normal state is a canonical one.[10] Of course, this is not true for non-commutative cases. However the space H constructed in the proof of Theorem 2 is a special Hilbert space on which the above-methioned Pallu de la Barrier's theorem remains true always.

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