17. On Ideals in Rings of Continuous Functions

By Taira SHIROTA
Osaka University, Osaka

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In the theory of function rings the boundedness of functions or the compactness of the base spaces plays a very important rôle, but it seems to be necessary to remove the condition of the boundedness or the compactness.

In the present note we concern ourselves with ideals of rings of continuous functions which are not always bounded.

1. **Definition 1.** Let X be a set. Then we say that a ring $\mathfrak{R}$ consisting of complex-valued functions on X with the ordinary addition and multiplication is a *normal function ring*, if it satisfies the following conditions:

(1) (*Self-adjointness*) It contains the identity and it contains a function with its conjugate function.

(2) (*Inverse closedness*) The subset $\mathfrak{R}_p$ of all strictly positive functions with inverses in $\mathfrak{R}$ possesses the following properties:

(a) If f and g belong to $\mathfrak{R}_p$, there exists an $h \in \mathfrak{R}_p$ such that $h \leqq f$ and $h \leqq g$.

(b) If f belongs to $\mathfrak{R}_p$, there exists a $g \in \mathfrak{R}_p$ such that $g^2 \leqq f$.

(c) If $f \geqq g$, $f \in \mathfrak{R}$ and $g \in \mathfrak{R}_p$, then f has an inverse.

(3) (*Regularity*) If $\underset{x \in A}{\text{l.u.b.}} |h(x)| \lneqq \underset{x \in B}{\text{g.l.b.}} |h(x)|$ for two subsets A and B of X and for some $h \in \mathfrak{R}$, then A and B are separated by a positive function h' of $\mathfrak{R}$, i.e., $h'(A) \equiv 0$ and $h'(B) \equiv 1$.

Then we see that the concept of a normal function ring is an extension of that of the inverse closed, regular, $*$-commutative, algebra with the identity. We have many examples of normal function rings. For instance, the ring of all complex-valued continuous functions on a completely regular space X: $\mathfrak{C}(X, K)$, the ring of all complex-valued uniformly continuous functions on a uniform space X: $\mathfrak{C}_u(X, K)$, and the ring of all complex-valued r-differentiable functions on an r-differentiable manifold M^r: $\mathfrak{C}_r(M^r)$ $(0 \leqq r \leqq \infty)$ are normal function rings.

2. We introduce a natural topology in a normal function ring.

Definition 2. Let $\mathfrak{R}$ be a normal function ring with base X and let $U_\pi = \{f \mid f \in \mathfrak{R} \ \& \ |f| < \pi\}$ for some $\pi \in \mathfrak{R}_p$ where $|f|$ is the function whose value at any point x of X is the absolute value of $f(x)$. Then the m-topology of $\mathfrak{R}$ is the one with a fundamental system of neighbourhoods of 0 $\{U_\pi \mid \pi \in \mathfrak{R}_p\}$.

Then from the conditions (1) and (2) of Definition 1 we obtain the following:

Lemma 1. *A normal function ring with m-topology is a Q-ring in the sense of I. Kaplansky,*[1] *i.e., the set of all functions with inverses is open. Furthermore the inverse of its element is continuous wherever it is defined.*

From Lemma 1 any maximal ideal M of a normal function ring is closed in m-topology and so the ideal of the intersection of maximal ideals is closed in m-topology.

E. Hewitt[2] considered the m-topology of the ring $\mathfrak{C}(X, R)$ of all real-valued continuous functions and proving Lemma 1 in this case, he conjectured the following Theorem 1.

Theorem 1.[3] *Let $\mathfrak{R}$ be a normal function ring. Then an ideal of $\mathfrak{R}$ is the intersection of all maximal ideals containing it if and only if it is m-closed, i.e., closed in m-topology.*

To prove Theorem 1 we use the following lemmas.

Lemma 2. *Let g be an element of $\mathfrak{R}$ such that $|g| \not> \pi$ for any π of $\mathfrak{R}_p$. Then for any sufficiently small $\pi \in \mathfrak{R}_p$ there exists an element f of $\mathfrak{R}$ having the following property:*

1) $|f-g|<\pi$
2) *$Z(g, \pi')$ and $X-Z(f)$ are separated by a function of $\mathfrak{R}$, where* $Z(g, \pi')=\{x \mid |g(x)| \leqq \pi'(x)\}$ *for some* $\pi \in \mathfrak{R}_p$.

Proof. Let $P=\{x \mid |g(x)| \geqq \pi(x)\}$. Then we may assume that $Z(g, \pi/2)$ and P are not void. Furthermore P and $Z(g, \pi/2)$ are separated, for let $h'=(g\cdot\bar{g}-(\pi/2)^2)/\pi^2$, then $h'(x) \leqq 0$ for $x \in Z(g, \pi/2)$ and $h'(x) \geqq 3/4$ for $x \in P$, hence by (3) of Definition 1 there exists a positive function $h \in \mathfrak{R}$ such that $h(P)\equiv 1$ and $h(Z(g, \pi/2))\equiv 0$. Furthermore we may assume that $0 \leqq h \leqq 1$, for $Z(h, 1/2)$ and P are separated by a positive function h_1 of $\mathfrak{R}$ by (3) of Definition 1, i.e., $h_1(Z(h, 1/2))\equiv 1$ and $h_1(P)\equiv 0$, then, by (2) of Definition 1, $h'=h\bar{h}/(h\bar{h}+h_1\bar{h}_1)$ is a desired function of $\mathfrak{R}$. Furthermore let $f=gh$. Then $f \in \mathfrak{R}$ and $|f-g|<\pi$ and $Z(f) \supset Z(h) \supset Z(g, \pi/2)$ and $Z(g, \pi/3)$ and $X-Z(g, \pi/2)$ are separated, hence if $\pi'=\pi/3$, then $X-Z(f)$ and $Z(g, \pi')$ are separated.

Lemma 3. *Let I be an ideal and f and g be elements of $\mathfrak{R}$ such that $Z(g) \supset Z(f, \pi)$ for some $\pi \in \mathfrak{R}_p$. Then $f \in I$ implies $g \in I$.*

Proof. Since I is an ideal, $Z(f, \pi/2)$ and $X-Z(f, \pi)$ are separated. Hence there exists a positive function h of $\mathfrak{R}$ such that $h(Z(f, \pi/2))\equiv 1$ and $h(X-Z(g))\equiv 0$. Let $f_1=f\bar{f}+h(\pi/2)^2$. Then $f_1 \in \mathfrak{R}_p$. Furthermore let $f_2=f\bar{f}f_1^{-1}$. Then $f_2 g=g$ and $f_2 \in I$, hence $g \in I$.

The proof of Theorem 1. We have only to prove the sufficiency. Suppose that an ideal I is m-closed and $g \notin I$. Then there exists $\pi_0 \in \mathfrak{R}_p$ such that if $|f-g|<\pi_0$, $f \notin I$. We may assume that

$|g| \not> \pi$ for any $\pi \in \mathfrak{R}_p$. Then by Lemma 1, there exist $g' \in \mathfrak{R}$ and $h \in \mathfrak{R}$ such that $|g-g'|<\pi_0$ and $h(Z(g, \pi'))\equiv 1$ and $h(X-Z(g'))\equiv 0$.

Now suppose that $f'h+f''=1$ for some $f'' \in I$ and $f' \in \mathfrak{R}$. Then $Z(g')\supset Z(f'', 1/2)$ hence by Lemma 2 $g' \in I$ which is a contradiction. Accordingly there exists a maximal ideal M such that $M\supset I$ and $M \ni h$.

But $M \not\ni g$. For $h(Z(g, \pi'))\equiv 1$ and $Z(g, \pi')$ and $Z(h, 1/2)$ are separated by h, hence there exists a positive function h' of $\mathfrak{R}$ such that $h'(Z(g, \pi'))\equiv 0$ and $h'(Z(h, 1/2))\equiv 1$. Then $Z(g, \pi')\subset Z(h')$, hence $h' \in M$, if $g \in M$. But $h+h'\geqq 1/2$ hence $h+h' \notin M$ and so $h \notin M$ which is a contradiction.

Corollary. *The m-closure of a prime ideal of a normal function ring is a maximal ideal.*

Proof. Let I be a prime ideal of a normal function ring $\mathfrak{R}$ and let M_1 and M_2 be two different maximal ideals containing I. Then there exist f_1 and f_2 such that $f_i \in M_i$ $i=1, 2$, and such that $f_1+f_2=1$. Hence $Z(f_1, 1/2)\frown Z(f_2, 1/2)=\phi$ and by (3) of Definition 1 there exist $h_i (i=1, 2)$ such that $h_i(X-Z(f_i, 1/2))\equiv 0$ and $h_i(Z(f_i, 1/3))\equiv 1$, hence $h_1\cdot h_2=0$ and $h_i \notin I$ $i=1, 2$, which is a contradiction. Then by Theorem 1 we obtain the corollary.

3. Applying Theorem 1 the theorem of Gelfand-Kolmogoroff[4] can be generalized as follows:

Theorem 2. *Let $\mathfrak{R}$ be a normal function ring with base X and let $\mathfrak{R}^*$ be the subring consisting of all bounded functions of $\mathfrak{R}$. Then $\mathfrak{R}^*$ is a normal function ring with base X. Furthermore let $\mathfrak{R}'$ be a normal function ring with base X such that $\mathfrak{R}\supset\mathfrak{R}'\supset\mathfrak{R}^*$. Then the structure spaces[5] of $\mathfrak{R}$ and $\mathfrak{R}'$ are homeomorphic.*

Proof. Let I be an m-closed ideal of $\mathfrak{R}$ and let $p(I)$ be the ideal of $\mathfrak{R}'$ such that it is the m-closure of $I\frown\mathfrak{R}'$ in $\mathfrak{R}'$. Conversely for any m-closed ideal I' of $\mathfrak{R}'$ let I'' be the set of all function $f \in \mathfrak{R}$ such that there exists a positive function $f' \in I'$ with the following property: $Z(f)\supset Z(f', \pi')$ for some $\pi' \in \mathfrak{R}'_p$. Let $q(I')=$the m-closure of the ideal I'' in $\mathfrak{R}$.

We show that for any m-closed ideal I of $\mathfrak{R}$, $I=qp(I)$. Let $f \in I$. Then by Lemma 2, there exists $g \in \mathfrak{R}$ such that $|f-g|<\pi$ and $Z(f, \pi')$ and $X-Z(g)$ are separated by $h \in \mathfrak{R}$, i.e., $1\geqq h\geqq 0$, $h(Z(f, \pi'))\equiv 0$ and $h(X-Z(g))\equiv 1$. Then $Z(h)\supset Z(f, \pi')$, hence $h \in I\frown\mathfrak{R}^*\subset I\frown\mathfrak{R}'$ and so $h \in p(I)$. On the other hand $Z(g)\supset Z(h, 1/2)$, hence $g \in p(I'')$, accordingly $f \in qp(I)$. Conversely let $f \in qp(I)$. Then for any $\pi \in \mathfrak{R}_p$ there exists g such that $Z(g)\supset Z(h, \pi')$ for some $h \in p(I)$ and for some $\pi' \in \mathfrak{R}'_p$ and such that $|f-g|<\pi$. Since $p(I)=$the m-closure of $I\frown\mathfrak{R}'$, there exists $h' \in I\frown\mathfrak{R}'$ such that $|h-h'|<\pi'/2$. Then $Z(h', \pi'/2)\subset Z(h, \pi')\subset Z(g)$. Hence by Lemma 2, $g \in I$ and so $f \in I$.

Thus the mapping p of the class of all m-closed ideals of $\mathfrak{R}$ into the class of all m-closed ideals of $\mathfrak{R}'$ is one-to-one and onto and furthermore order-preserving, where the order is that of set-inclusion. This means that the structure spaces of $\mathfrak{R}$ and $\mathfrak{R}'$ are homeomorphic by Theorm 1.

Corollary. *Under the same assumption of Theorem 2, any closed ideal of $\mathfrak{R}'$ is the intersection of all closed primary ideals containing it under the relative topology of $\mathfrak{R}'$ induced by the m-topology of $\mathfrak{R}$. Furthermore if the m-topology of $\mathfrak{R}'$ and the above relative topology are different, then there exists a closed ideal under the relative topology which is not the intersection of closed maximal ideals containing it.*

For we have only to consider the ideal $I=\{g|$ for any $\pi\in\mathfrak{R}_p$ there exists $\pi'\in\mathfrak{R}'_p$ such that $Z(g,\pi)\supset Z(\pi_0,\pi')\}$, where $\pi_0\in\mathfrak{R}_p$ and $\pi_0\not>\pi'$ for any $\pi'\in\mathfrak{R}'_p$.

4. Definition 3. Let $\mathfrak{B}=\{Y\}$ be a family of subsets of X with following properties: i) if Y_1 and Y_2 are contained in $\mathfrak{B}$, then $Y_1\cup Y_2\in\mathfrak{B}$ and ii) the sum of all Y of $\mathfrak{B}$ is X itself.

Furthermore let $\mathfrak{R}$ be a normal function ring with base X and let $U\pi,_Y=\{g|\,|g(x)|<\pi(x)$ on $Y\}$. Then $(m,\mathfrak{B})$-topology of $\mathfrak{R}$ is the one with a fundamental system of neighbourhoods of $0\ \{U\pi,_Y|\pi\in\mathfrak{R}_p\ \&\ Y\in\mathfrak{B}\}$.

Then we have the following

Theorem 3. *Any normal function ring with the $(m,\mathfrak{B})$-topology is a topological ring in which any closed ideal is the intersection of closed maximal ideals containing it.*

Proof. Let I be a closed ideal of $\mathfrak{R}$ and let g be an element of $\mathfrak{R}$ which is not contained in I. Then there exist $\pi\in\mathfrak{R}_p$, and $Y\in\mathfrak{B}$ such that if $|g-g'|<\pi$ on Y, $g'\notin I$.

Now let $\mathfrak{R}'$ be the ring consisting of functions of $\mathfrak{R}$ confined to Y. Then $\mathfrak{R}'$ satisfies the conditions (1) and (3) of Definition 1. Furthermore let $\mathfrak{R}'_{p'}=\{f|Y|f\in\mathfrak{R}_p\}$. Then $\mathfrak{R}'_{p'}$ satisfies the condition (2) of Definition 1 and if $f\in\mathfrak{R}'_{p'}$, then $f^2\in\mathfrak{R}'_{p'}$ and $f/n\in\mathfrak{R}'_{p'}$. Hence by the same method used in the proof of Theorem 1 there exists a maximal ideal M' of $\mathfrak{R}'$ such that $M'\not\ni g|Y$ and $M'\supset I'$, where $I'=\{f|Y|f\in I\}$. Hence $M=\{f|f|Y\in M'\}$ is the maximal ideal containing I and not containing g. Furthermore M is closed in our topology.

References

1) Cf. I. Kaplansky: Topological ring, Amer. J. Math., **69** (1947).

2) Cf. E. Hewitt: Rings of real-valued continuous functions I, Trans. Amer. Math., **64** (1948).

3) Recently I have been informed from Professor E. Hewitt that L. Gillman, M. Henriksen, and M. Jerison proved the theorem 1 in the case of $\mathfrak{R}=\mathfrak{C}(X,R)$. But

their paper is yet unavailable to the present author.

4) Cf. I. Gelfand and A. N. Kolmogoroff: On rings of continuous functions on topological spaces, C. R. Acad. Sci., URSS, **22** (1939).

5) Cf. M. H. Stone: Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., **41** (1937).

Cf. I. Gelfand and G. Šilov: Über verschiedene Methoden der Einführung der Topologie in der Menge der maximalen Idealen eines normierten Ringes, Rec. Math. (Mat. Sbornik) N. S., **9** (1941).