16. On Quasi-Translations in E^n

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By a *quasi-translation* will be meant a sense preserving topological transformation f of a Euclidean space E^n onto itself such that for every bounded set M its iterated images $f^n(M)$ for $n\to\pm\infty$ have no cluster set, i.e.

$$\overline{\lim}_{n\to\pm\infty} f^n(M)=0,$$

or roughly speaking, $f^n(M)$ diverges to infinity when $n\to\pm\infty$.

A quasi-translation is a fortiori fixed point free and moreover regular (or singularity free) in the sense of Kerékjártó-Sperner. Thus a quasi-translation is by the theorem of Kerékjártó-Sperner[1][2] topologically equivalent to a translation in the ordinary sense if E^n is a plane. Whether or not this is true for $n\geqq 3$ remains still open. The purpose of this note is to give a simple proof of Theorem I, which may serve as a lemma to settle this question. The theorem of Kerékjártó-Sperner is an immediate consequence of our theorem.

Theorem I. *Let f be a quasi-translation of E^n. Then there is an unbounded polyhedron π such that if D denotes the domain bounded by π and $f(\pi)$, then $f^n(D)$ is disjoint from $f^m(D)$ whenever $n\neq m$, n and m being arbitrary integers, and $\bigcup_{n=-\infty}^{\infty} f^n(\overline{D})=E^n$.*

We prove the theorem in the following version, in which the sense preservation is not even assumed.

Theorem II. *Let f be a topological transformation of a sphere S^n onto itself with a single fixed point o such that if M is a set with $\overline{M}\not\ni o$, then*

$$\lim_{n\to\pm\infty} f^n(M)=o.$$

Then there exists an open polyhedron π with the sole boundary at o such that if D denotes the domain bounded by $\pi\smile o$) and $f(\pi\smile o)$, then $f^n(D)$ is disjoint from $f^m(D)$ whenever $n\neq m$, n and m being arbitrary integers, and $\bigcup_{n=-\infty}^{\infty} f^n(\overline{D})=S^n$.*

Proof. To begin with, we shall define for any set M of S^n the measure $\mu(M)$ introduced by H. Whitney[3] as follows: Let a_1, a_2, $\dots$, $a_n,\dots$ be a sequence of points dense in S^n, and put for any

*) o denotes the point o as well as the set consisting of the point o. $\pi\smile o$ means the set sum of π and o.

point x of S^n

$$f_n(x)=\frac{1}{1+d(x,a_n)}.^{**)}$$

Given a set M, let

$$\mu_n(M)=\sup_{x\in M} f_n(x)-\inf_{x\in M} f_n(x)$$

and let

$$\mu(M)=\sum_{n=1}^{\infty}\frac{\mu_n(M)}{2^n}.$$

Then $\mu(M)$ is defined for every set M of S^n and we have

W_1. $0\leqq\mu(M)\leqq d(M)$.**)

W_2. If $M\subset N$, then $\mu(M)\leqq\mu(N)$.

W_3. If $M\subset U(N;\varepsilon)$,**) then $\mu(M)<\mu(N)+\varepsilon$.

W_4. If $M\subset N$ and if N contains at least one point which has a positive distance from M, then $\mu(M)<\mu(N)$ (Whitney[3]).

In the following we shall make free use of these properties W_1-W_4 of Whitney's μ-measure.

For every point x of S^n consider the set

$$\bigcup_{n=0}^{\infty} f^n(x)=\{f^n(x)\mid n\geqq 0\}$$

where $f^0(x)$ and $f^1(x)$ stand for x and $f(x)$ respectively, and correspondingly the function

$$\mu(\bigcup_{n=0}^{\infty} f^n(x))=g_+(x).$$

Then $g_+(x)$ is continuous at every point x except at $x=o$. For, given a positive number ε, there can be found a neighbourhood U of x such that $d(f^n(U))<\varepsilon$ for all $n\geqq 0$ by the continuity of f and by the hypothesis of regularity that

$$\lim_{n\to\infty} f^n(U)=o$$

whenever $\bar{U}\not\ni o$. Then for every point $y\in U$

$$f^n(y)\subset U(\bigcup_{n=0}^{\infty} f^n(x);\varepsilon)$$

and

$$f^n(x)\subset U(\bigcup_{n=0}^{\infty} f^n(y);\varepsilon)$$

hold and hence by W_3

$$|\mu(\bigcup_{n=0}^{\infty} f^n(x))-\mu(\bigcup_{n=0}^{\infty} f^n(y))|<\varepsilon,$$

whence the continuity of $g_+(x)$ at $x\neq o$ follows.

Next put

$$g_-(x)=\mu(\bigcup_{n=0}^{-\infty} f^n(x)).$$

Then $g_-(x)$ is likewise continuous at x except at $x=o$ and so is the function

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**) $d(a,b)$, $d(M)$ and $U(M;\varepsilon)$ are the distance between a and b, the diameter of M and the ε-neighbourhood of M respectively on S^n.

$$\varphi(x)=g_+(x)-g_-(x).$$

Now take a point p fixed and different from o. Then, if $n>0$ is taken sufficiently large, $g_+(f^n(p))$ can be made as small as we please, while

$$g_-(f^n(p))=\mu(\bigcup_{i=n}^{-\infty} f^i(p))$$
$$>\mu(\bigcup_{i=0}^{-\infty} f^i(p))=g_-(p)>0,$$

so that $\varphi(f^n(p))=g_+(f^n(p))-g_-(f^n(p))$ becomes negative. By the same reason $\varphi(f^n(p))$ becomes positive if $n<0$ is chosen large enough in absolute value. It follows from the continuity of f that there must also be points x with $\varphi(x)=0$ other than o. If we put therefore

$$\Phi_0=\{x\,|\,\varphi(x)=0\},$$
$$\Phi_+=\{x\,|\,\varphi(x)>0\},$$
$$\Phi_-=\{x\,|\,\varphi(x)<0\},$$

then Φ_0, Φ_+ and Φ_- are all non void.

Since $\varphi(x)=0$ for $x\neq o$ implies $\varphi(f(x))<0$ by the definition of $\varphi(x)$ and on account of W_4, we have

$$\Phi_0\frown f(\Phi_0)=0.$$

Moreover we have

$$f(\Phi_-)\subset\Phi_-.$$

Now let U be a domain such that $\bar{U}\not\ni o$ and $U\frown f(U)\neq 0$. Then for every n

$$(1)\qquad f^n(U)\frown f^{n+1}(U)\neq 0,$$

and since $\bar{U}\not\ni o$, there is a positive number d such that for every $y\in U$

$$\mu(\bigcup_{i=0}^{-\infty} f^i(y))>d>0.$$

But given a positive number ε there is by the hypothesis on f a positive number N such that

$$f^n(U)\subset U(o;\varepsilon)$$

for all $n\geqq N$. Therefore, if ε is chosen $<d$, then for any $x\in f^n(U)$ and for any $n\geqq N$ we have, since $f^{-n}(x)\in U$,

$$\varphi(x)=\mu(\bigcup_{i=0}^{\infty} f^i(x))-\mu(\bigcup_{i=0}^{-\infty} f^i(x))$$
$$<\mu(\bigcup_{i=0}^{\infty} f^i(x))-\mu(\bigcup_{i=-n}^{-\infty} f^i(x))$$
$$<\varepsilon-d$$
$$<0,$$

which indicates that all $f^n(U)$ are contained in Φ_- if $n\geqq N$. If we denote by D_- the component of Φ_- which contains $f^N(U)$, then $f^n(U)$ is wholly contained in D_- whenever $n\geqq N$, in consequence of the relation (1).

Since the boundary of Φ_- is evidently contained in Φ_0, every boundary point of D_- is also a point of Φ_0.

$f(D_-)$ is wholly contained in D_-. For first, since D_- and $f(D_-)$ have the set $f^{N+1}(U)$ in common, they intersect. Second, if there were a point x of $f(D_-)$ outside D_-, connect x and a point q of $f^{N+1}(U)$ by an arc within $f(D_-)$. Then it must intersect the boundary $\dot{D}_-$ of D_- and thus there would exist a point r of $\dot{D}_-$ in $f(D_-)$, which is absurd, since $r \in \dot{D}_- \subset \Phi_0$ but $f(D_-) \frown \Phi_0 = 0$.

Now let $\{U_i\}$ be a covering of $S^n - o$ consisting of a countable number of domains U_i such that $\overline{U}_i \not\ni 0$ and $U_i \frown f(U_i) \neq 0$, and corresponding to each U_i let D_i be the component of Φ_- described above, that is the component of Φ_- with the property that $f^n(U_i)$ are all contained in D_i if $n \geqq N_i$ for some natural number N_i. We assert that in reality D_i all coincide.

To prove this, suppose the contrary were the truth, and changing suitably the suffixes of D_i if necessary, let $D_1, D_2, \ldots, D_i, \ldots$ $(2 \leqq i < \infty)$ be the finite or infinite sequence of all distinct D_i. Then, if p is any point of $S^n - o$, there is an element of $\{U_i\}$, say U_i, which contains p, but, since D_i contains by its definition $f^{N_i}(U_i)$, p is contained in $f^{-N_i}(D_i)$. Consequently we have

$$(2) \qquad \bigcup_{n=-\infty}^{\infty} \bigcup_{i=1}^{\infty} f^n(D_i) = S^n - o.$$

On the other hand, since D_i are disjoint, we have

$$f^n(D_i) \frown f^n(D_j) = 0$$

for every n whenever $i \neq j$. But since $D_i \supset f(D_i)$, we have

$$f^n(D_i) \frown f^m(D_j) = 0$$

for any integers n and m. Thus by (2) $S^n - o$ is seen to be expressed as the sum of at least two, and at most a countably infinite number of, disjoint domains

$$\bigcup_{n=-\infty}^{\infty} f^n(D_i),$$

which is absurd. Therefore all D_i must coincide, and each D_i is nothing other than D_- we have considered above.

Thus we have obtained the following result:

Under the hypothesis on f of Theorem II there exists a domain $D_- \subset \Phi_-$ such that

$$(3) \qquad D_- \supset f(D_-),\ \mathring{D}_- \frown f(\dot{D}_-) = 0 \quad \textit{and} \quad \bigcup_{n=-\infty}^{\infty} f^n(D_-) = S^n - o.$$

By covering D_- in the usual way with a family of cubes which intersect D_- but which are disjoint from $f^{-1}(\dot{D}_-)$, we can obtain from D_- a domain P bounded by one or more of open polyhedra with the sole boundary at o such that

$$D_- \subset P \supset f^{-1}(D_-).$$

Proceeding exactly as above we can obtain analogous to D_- a component D_+ of Φ_+ such that (3), D_- substituted by D_+, holds true. Now, if the boundary of P consists of more than one component, let π be that component which can be joined by an arc j to a point of D_+ outside P. Then π is obviously the required polyhedron.

References

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3) H. Whitney: Regular families of curves, Ann. Math., **34**, 244–270 (1933).