15. On the Structure of Algebraic Systems

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The structure of an algebraic system A has been discussed by K. Shoda[2)3)4)] under the following conditions:

SI. *A has a null-element.*

SII. *The subsystem generated by any two normal subsystems of A is normal in A.*

SIII. *The meromorphism of any two algebraic systems which are homomorphic to A is always class-meromorphism.*

(SII *and* SIII *are assumed for any subsystem of A.*)

G. Birkhoff has introduced in his book[1)] the following condition which is equivalent to SIII: *all congruences on A are permutable.*

In the present paper we shall give a new definition of normal subsystems, and study on the normal subsystems and the congruences of an algebraic system A (§1). Moreover under weaker conditions than SII, SIII (§2), we shall discuss the Jordan-Hölder-Schreier theorem (§3) and the Remak-Schmidt-Ore theorem for A (§4).

§1. Normal Subsystems and Congruences. Throughout this paper we put the following conditions on the algebraic system A to keep out the complication.

0. *All compositions are binary and single valued, moreover any two elements may be composable by any composition.*

I. *A has a null-element e ($e\alpha e=e$ for any composition α).*

A subset B of A is called a *subsystem* if B is closed under any composition of A and contains e.

Let $f(\xi_1, \dots, \xi_n)$ be a polynomial by compositions of A. In the following $f(X, x_2, \dots, x_n)$ denotes the set $\{f(x, x_2, \dots, x_n) : x \in X\}$, where $X \subset A$, $x_2, \dots, x_n \in A$. Then $f(X, x_2, \dots, x_n)$ is of course a subset of A.

Definition 1. *A subset C is called a coset if and only if the following condition holds for any polynomial $f(\xi_1, \dots, \xi_n)$ and any elements $x_2, \dots, x_n \in A$,*

$$f(C, x_2, \dots, x_n) \frown C \neq \phi \quad \textit{implies} \quad f(C, x_2, \dots, x_n) \subset C.$$

A coset C is called a normal subsystem, when C forms a subsystem of A.

Theorem 1. *Any coset C is a residue class of a congruence and conversely.*

Proof. We define $a \overset{\theta}{\sim} b$ $(a, b \in A)$ when there exist polynomials $f_1, \dots, f_n$ such that

$$a \in f_1(C, x_{1,2}, \dots, x_{1,r_2}),$$
$$f_1(C, x_{1,2}, \dots, x_{1,r_1}) \frown f_2(C, x_{2,2}, \dots, x_{2,r_2}) \neq \phi,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$f_{n-1}(C, x_{n-1,2}, \dots, x_{n-1,r_{n-1}}) \frown f_n(C, x_{n,2}, \dots, x_{n,r_n}) \neq \phi,$$
$$f_n(C, x_{n,2}, \dots, x_{n,r_n}) \ni b.$$

It is easily verified that the relation θ is a congruence on A. Since $C \subset f(C)$ for the polynomial $f(\xi)=\xi$, C is contained in a class C' corresponding to the congruence θ. Then $C=C'$ follows from the definitions of θ and the coset. Therefore θ is a congruence having a giving coset C as a class. The converse follows from the property of the congruence.

Remark. In order that any two elements of C are to be congruent by a congruence φ, it is necessary that φ has the property of θ. Hence the congruence θ is the least one corresponding to the coset C.

K. Shoda and G. Birkhoff have defined normal subsystems as follows: *The residue class of A with respect to a congruence which contains e is called a normal subsystem of A.* This definition is evidently equivalent to our definition.

In the following θ_L denotes the congruence of a subsystem L, $\theta_L(B)$ the congruence naturally induced by θ_L on $B \subset L$. B/θ_L denotes the residue class system of B with respect to $\theta_L(B)$. For a subset $M \subset B$, we denote by $(M|\theta_L(B))$ the set $\{x : x \overset{\theta_L(B)}{=\!=} m \in M\}$. If M is a subsystem of B, then the set $(M|\theta_L(B))$ forms clearly a subsystem of B.

Theorem 2. *Let C be any subsystem such that $(B|\theta_L) \supset C \supset B$. Then C/θ_L and B/θ_L are isomorphic.*

Proof. Let $\{C_i\}$ be the set of all cosets of C corresponding to $\theta_L(C)$, and let $B_i = C_i \frown B$, then $\{B_i\}$ is the set of all the cosets of B corresponding to $\theta_L(B)$. And the correspondence $C_i \longleftrightarrow B_i$ is evidently an isomorphism of C/θ_L and B/θ_L.

Theorem 3. *Let $L \supset B \supset C$, $M \supset C$, and $\theta_L(C) \leqq \varphi_M(C)$. Then the congruence $\varphi_M(C)$ can be extended on the subsystem $(C|\theta_L(B))$.*

Proof. Let $a, b \in (C|\theta_L(B))$. We define $a \sim b$ if and only if there exist $x_1, x_2 \in C$ such that $a \overset{\theta_L(B)}{=\!=} x_1 \overset{\varphi_M(C)}{=\!=} x_2 \overset{\theta_L(B)}{=\!=} b$. Then it is easily verified that the relation $\sim$ is a congruence on $(C|\theta_L(B))$. And by $\theta_L(C) \leqq \varphi_M(C)$ the congruence $\sim$ is equivalent to $\varphi_M(C)$ on C.

In the following we denote by $[\varphi_M(C)|\theta_L(B)]$ the extended congruence of $\varphi_M(C)$ as in Theorem 3.

Definition 2. *Let N be a normal subsystem of A. The congruence*

θ is called a lower-congruence if θ is a least congruence corresponding to N.

The lower-congruence corresponding to N may be constructed as in the proof of Theorem 1. Hence we can easily prove the following:

Theorem 4. *The join of any two lower-congruences is a lower-congruence.*

§2. **Conditions for Algebraic Systems.** In order to extend our theory, we discuss some conditions for the algebraic system A.

Let $\theta(B)$ be a congruence on a subsystem B. We denote by $S_x(\theta(B))$ the coset containing $x \in B$ which corresponds to $\theta(B)$, and for convenience, by $S(\theta(B))$ the normal subsystem $S_e(\theta(B))$.

We consider the following conditions:

II. $(S(\theta)|\varphi)=(S(\varphi)|\theta)$ *for any lower-congruences θ, φ on A.*

II*. $(S(\theta_L(L\frown M))|\varphi_M(L\frown M))=(S(\varphi_M(L\frown M))|\theta_L(L\frown M))$ *for any lower-congruences θ_L and φ_M on L and M respectively, where L, M are any subsystems which appear in normal chains of A.*

III. $(S(\theta)|\varphi)=(S_x(\varphi)|\theta)$ *for an element x satisfying $S(\theta)\frown S_x(\varphi)\neq\phi$, where θ and φ are lower-congruences on A.*

Using our notations we can describe the condition SIII as follows:

$$(S_x(\theta)|\varphi)=(S_y(\varphi)|\theta) \text{ for } x,\ y \text{ satisfying } S_x(\theta)\frown S_y(\varphi)\neq\phi.$$

Then the condition SIII* that any subsystem B of A satisfies the condition SIII can be described as follows:

$$(S_x(\theta(B))|\varphi(B))=(S_y(\varphi(B))|\theta(B)) \text{ for } x,\ y \text{ satisfying}$$
$$S_x(\theta(B))\frown S_y(\varphi(B))\neq\phi.$$

By the above descriptions of the conditions we can easily see the following implications:

$$\text{SIII*}\rightarrow\text{SIII}\rightarrow\text{III}\rightarrow\text{II and SIII*}\rightarrow\text{II*}\rightarrow\text{II}.$$

We now prove the

Theorem 5. *The condition* SII *implies* II.

Proof. Let θ, φ be any lower-congruences on A. Let ω be the lower-congruence corresponding to the subsystem B generated by $S(\theta)$ and $S(\varphi)$. Since the subsystem $(S(\theta)|\varphi)$ contains $S(\theta)$ and $S(\varphi)$, $S(\omega)=B\subset(S(\theta)|\varphi)$. On the other hand, $S(\omega)\supset S(\varphi)$ implies $\omega\geqq\varphi$, and therefore $S(\omega)\supset(S(\theta)|\varphi)$. Hence $S(\omega)=(S(\theta)|\varphi)$. Similarly $S(\omega)=(S(\varphi)|\theta)$. Therefore we get $(S(\theta)|\varphi)=(S(\varphi)|\theta)$.

Theorem 6. *If A has the condition* II, *then the set $\mathfrak{L}_A$ of all the normal subsystems of A forms a modular lattice under meet, the intersection and join $S(\theta)\smile S(\varphi)=(S(\theta)|\varphi)$ where θ, φ are lower-congruences on A.*

Proof. Let θ, φ be lower-congruences. Then $S(\theta)\smile S(\varphi)=S(\theta\smile\varphi)$, since $S(\theta)\smile S(\varphi)=(S(\theta)|\varphi)=(S(\varphi)|\theta)$. By the definition we have $S(\theta)\frown S(\varphi)=S(\theta\frown\varphi)$. $\theta\frown\varphi$ is not always a lower-congruence, we denote by $\overline{\theta\frown\varphi}$ the lower-congruence corresponding to $S(\theta\frown\varphi)$. Then

$(S(\theta)\frown S(\varphi))\smile S(\theta)=S(\theta\frown\varphi)\smile S(\theta)=S(\overline{\theta\frown\varphi})\smile S(\theta)=S(\overline{(\theta\frown\varphi)}\smile\theta)=S(\theta)$. Since $\theta\smile\varphi$ is a lower-congruence, $(S(\theta)\smile S(\varphi))\frown S(\theta)=S(\theta\smile\varphi)\frown S(\theta)=S((\theta\smile\varphi)\frown\theta)=S(\theta)$. Hence $\mathfrak{L}_A$ forms a lattice.

Let θ, φ_i be lower-congruences. Let $S(\varphi_1)\subsetneqq S(\varphi_2)$ and $S(\varphi_1)\smile S(\theta)=S(\varphi_2)\smile S(\theta)$. From $S(\varphi_1)\subsetneqq S(\varphi_2)$ we get $\varphi_1<\varphi_2$. Then $(S(\theta)|\varphi_1)/\varphi_1=S(\varphi_1)\smile S(\theta)/\varphi_1=S(\varphi_2)\smile S(\theta)/\varphi_1\supset S(\varphi_2)/\varphi_1$. Hence $S(\varphi_2)/\varphi_1$ contains at least one coset different from $S(\varphi_1)$. Hence $S(\varphi_1)\frown S(\theta)\subsetneqq S(\varphi_2)\frown S(\theta)$. Therefore $\mathfrak{L}_A$ is modular.

§3. **Normal Chains.** We denote by $B//N$ the residue class system of $B(\subset A)$ with respect to the greatest congruence corresponding to a normal subsystem N of B.

Theorem 7. (Schreier theorem for normal chains) *If A has the condition* II*, *then any two finite normal chains*

(1) $\quad A=A_0\supset S(\theta_0(A_0))=A_1\supset\cdots\supset S(\theta_{n-1}(A_{n-1}))=A_n=e,$

(2) $\quad A=B_0\supset S(\varphi_0(B_0))=B_1\supset\cdots\supset S(\varphi_{m-1}(B_{m-1}))=B_m=e$

can be refined by interpolation of terms $A_{i,j}=(A_i\frown B_j|\theta_i(A_i))$ and $B_{i,j}=(A_i\frown B_j|\varphi_j(B_j))$ such that $A_{i,j}//A_{i,j+1}$ and $B_{i,j}//B_{i+1,j}$ are isomorphic, where $\theta_i(A_i),\varphi_j(B_j)$ are lower-congruences on A_i, B_j respectively.

Proof. Let $\omega(A_i\frown B_j)=\theta_i(A_i\frown B_j)\smile\varphi_j(A_i\frown B_j)$. Then $[\omega(A_i\frown B_j)|\theta_i(A_i)]$ is just defined on $(A_i\frown B_j|\theta_i(A_i))$. And we get

$$S([\omega(A_i\frown B_j)|\theta_i(A_i)])=(S(\omega(A_i\frown B_j))|\theta_i(A_i))$$
$$=(S(\theta_i(A_i\frown B_j)\smile\varphi_j(A_i\frown B_j))|\theta_i(A_i))=((S(\varphi_j(A_i\frown B_j))|\theta_i(A_i\frown B_j))|\theta_i(A_i))$$
$$=(S(\varphi_j(A_i\frown B_j))|\theta_i(A_i))=(A_i\frown S(\varphi_j(B_j))|\theta_i(A_i))$$
$$=(A_i\frown B_{j+1}|\theta_i(A_i))=A_{i,j+1}.$$

Hence (1) is refined by interpolation of terms $A_{i,j}$. Similarly (2) is refined by interpolation of terms $B_{i,j}$.

Since $(A_i\frown B_j|[\omega(A_i\frown B_j)|\theta_i(A_i)])=(A_i\frown B_j|\theta_i(A_i))$, we get

$$(A_i\frown B_j|\theta_i(A_i))/[\omega(A_i\frown B_j)|\theta_i(A_i)]$$
$$\cong A_i\frown B_j/[\omega(A_i\frown B_j)|\theta_i(A_i)]=A_i\frown B_j/\omega(A_i\frown B_j).$$

Similarly $(A_i\frown B_j|\varphi_j(B_j))/[\omega(A_i\frown B_j)|\varphi_j(B_j)]\cong A_i\frown B_j/\omega(A_i\frown B_j)$. Hence $A_{i,j}/[\omega(A_i\frown B_j)|\theta_i(A_i)]\cong B_{i,j}/[\omega(A_i\frown B_j)|\varphi_j(B_j)]$. Therefore $A_{i,j}//A_{i,j+1}\cong B_{i,j}//B_{i+1,j}$.

Remark. The Schreier theorem for normal chains consisting of normal subsystems follows from the modularity of the lattice $\mathfrak{L}_A$.

§4. **Direct Decompositions.** In the following we assume that A has not only the conditions 0 and I but also III.

Theorem 8. *Let θ, φ be any lower-congruences on A such that $\theta\frown\varphi=0$, then $S(\theta\smile\varphi)$ and $S(\theta)\times S(\varphi)$ are isomorphic.*

Proof. For any element x in $S(\theta\smile\varphi)$, we denote by $S_x(\theta)$ the coset of $S(\theta\smile\varphi)/\theta$ containing x, and by $S_x(\varphi)$ the coset of $S(\theta\smile\varphi)/\varphi$ containing x. Then the correspondence $x\to(S_x(\varphi),S_x(\theta))$ is a homomorphism of $S(\theta\smile\varphi)$ onto a subsystem B of $S(\theta\smile\varphi)/\varphi\times S(\theta\smile\varphi)/\theta$. Since $\theta\frown\varphi=0$, the homomorphism is an isomorphism. By the condition III, we get $S_x(\varphi)\frown S_y(\theta)\neq\phi$ for any x, y in $S(\theta\smile\varphi)$. Hence $B=S(\theta\smile\varphi)/\varphi\times S(\theta\smile\varphi)/\theta$. Using $\theta\frown\varphi=0$, we get easily

$$S(\theta\smile\varphi)/\varphi=(S(\theta)|\varphi)/\varphi\cong S(\theta)/\theta\frown\varphi=S(\theta).$$

Similarly $S(\theta\smile\varphi)/\theta\cong S(\varphi)$. Hence $S(\theta\smile\varphi)$ and $S(\theta)\times S(\varphi)$ are isomorphic.

Theorem 9. *Let $A=S(\theta_1)\smile\cdots\smile S(\theta_n)$ be any representation of A as a direct join decomposition in the lattice $\mathfrak{L}_A$. Then A is isomorphic to $S(\theta_1)\times\cdots\times S(\theta_n)$ if A has the condition* (*): *$S(\theta)=e$ implies $\theta=0$.*

Proof. There exist lower-congruences θ'_i such that $S(\theta'_i)=S(\theta_i)$. Putting θ'_i in place of θ_i, we get by the assumption $(S(\theta'_1)\smile\cdots\smile S(\theta'_{i-1}))\frown S(\theta'_i)=e$. Hence $S(\theta'_1\smile\cdots\smile\theta'_{i-1})\frown S(\theta'_i)=e$, $S((\theta'_1\smile\cdots\smile\theta'_{i-1})\frown\theta'_i)=e$. Using the condition (*), we get $(\theta'_1\smile\cdots\smile\theta'_{i-1})\frown\theta'_i=0$. Hence by Theorems 4 and 8, we get $S(\theta'_1\smile\cdots\smile\theta'_i)\cong S(\theta'_1\smile\cdots\smile\theta'_{i-1})\times S(\theta'_i)$. Therefore $A\cong S(\theta_1)\times\cdots\times S(\theta_n)$.

Theorem 10. *Let $A\cong A_1\times\cdots\times A_n$ be any representation of A as a direct product. If A has not an infinite normal chain, then there exist lower-congruences $\theta_1,\dots,\theta_n$ such that $S(\theta_i)\cong A_i$ and $A=S(\theta_1)\smile\cdots\smile S(\theta_n)$ is a direct join decomposition in the lattice $\mathfrak{L}_A$.*

Proof. We denote by $A\ni a\sim(a_1,\dots,a_n)\in A_1\times\cdots\times A_n$ the correspondence of the isomorphism of A and $A_1\times\cdots\times A_n$. We define $(a_1,\dots,a_n)\sim a\overset{\theta'_i}{\equiv}b\sim(b_1,\dots,b_n)$ when $a_k=b_k$ for $k\neq i$. Then θ'_i is a congruence on A such that $S(\theta'_i)\cong A_i$. Let $\theta_1,\dots,\theta_n$ be lower-congruences such that $S(\theta_i)=S(\theta'_i)$, then $A\cong S(\theta_1)\times\cdots\times S(\theta_n)$. Since $\theta'_1,\dots,\theta'_n$ are independent, $\theta_1,\dots,\theta_n$ are independent. Hence by Theorems 4 and 8, $S(\theta_1\smile\cdots\smile\theta_i)\cong S(\theta_1\smile\cdots\smile\theta_{i-1})\times S(\theta_i)$. Therefore $S(\theta_1\smile\cdots\smile\theta_n)\cong S(\theta_1)\times\cdots\times S(\theta_n)\cong A$. If $S(\theta_1\smile\cdots\smile\theta_n)\subsetneqq A$, then there exists an infinite normal chain of A. This contradicts the assumption of this theorem. Hence $A=S(\theta_1\smile\cdots\smile\theta_n)=S(\theta_1)\smile\cdots\smile S(\theta_n)$. And it is evident that $A=S(\theta_1)\smile\cdots\smile S(\theta_n)$ is a direct join decomposition in the lattice $\mathfrak{L}_A$.

Theorem 11. (Remak-Schmidt-Ore theorem for direct join decompositions) *Let $A=S(\theta_1)\smile\cdots\smile S(\theta_n)=S(\varphi_1)\smile\cdots\smile S(\varphi_m)$ be any two representations as a direct join decomposition of indecomposable factors in the lattice $\mathfrak{L}_A$. If (i) A has the condition* (*), *(ii) $\mathfrak{L}_A$ has finite length, then $n=m$, and $S(\theta_i)$, $S(\varphi_j)$ are pairwise isomorphic, moreover $S(\theta_i)$ and $S(\varphi_j)$ are mutually replaceable.*

Proof. By the modularity of $\mathfrak{L}_A$, we get that $n=m$, and $S(\theta_i)$, $S(\varphi_j)$ are pairwise projective, moreover $S(\theta_i)$ and $S(\varphi_i)$ are mutually replaceable. Assuming that $\theta_1,\dots,\theta_n$, $\varphi_1,\dots,\varphi_m$ are lower-congruences without loss of generality, we get that $\theta_i^*=\theta_1\smile\cdots\smile\theta_{i-1}\smile\theta_{i+1}\smile\cdots\smile\theta_n$ is a lower-congruence. Hence $A/\theta_i^*=(S(\theta_i)|\theta_i^*)/\theta_i^*\cong S(\theta_i)/\theta_i^*=S(\theta_i)/\theta_i\smile\theta_i^*=S(\theta_i)$. Similarly $A/\theta_i^*=(S(\varphi_j)|\theta_i^*)/\theta_i^*\cong S(\varphi_i)/\theta_i^*=S(\varphi_j)/\varphi_j\frown\theta_i^*=S(\varphi_j)$. Therefore $S(\theta_i)\cong S(\varphi_j)$.

Theorem 12. (Remak-Schmidt-Ore theorem for direct product decompositions) *Let* $A \cong A_1 \times \cdots \times A_n \cong B_1 \times \cdots \times B_m$ *be any two representations as a direct product of indecomposable factors. If* (*i*) A *has the condition* (*), (*ii*) A *has no infinite normal chain, then* $n=m$, *and* A_i, B_j *are pairwise isomorphic.*

Proof. This theorem is immediate by Theorems 9,10 and 11.

References

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