33. *Probabilities on Inheritance in Consanguineous Families. V*

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V. Mother-descendant combinations through a single consanguineous marriage

1. Mother-descendant combination immediate after a consanguineous marriage

Up to the last chapter, any consanguineous marriage has never been implicated. We now begin to attack the problems concerning a consanguineous marriage.

Let μth and νth descendants collaterally originated from a mother $A_{\alpha\beta}$ and her same spouse be married consanguineously and then originate themselves an nth descendant $A_{\xi\eta}$. Our present purpose is to determine the probability of combination $(A_{\alpha\beta};\ A_{\xi\eta})$ which will be designated by

$$\pi_{\mu\nu;n}(\alpha\beta;\ \xi\eta)\equiv\overline{A}_{\xi\eta}\kappa_{\mu\nu;n}(\alpha\beta;\ \xi\eta).$$

We distinguish *three systems* according to $\mu=\nu=1$, $\mu>1=\nu$ or $\mu=1<\nu$, and $\mu,\ \nu>1$. However, the final results for $\pi_{\mu\nu;n}$ will be, contrary to $\pi_{\mu\nu}$ discussed in III, unified into a unique expression for any pair of μ, ν with $\mu\geqq1$, $\nu\geqq1$.

We first deal with *the case* $n=1$. Its defining equation given by

$$\kappa_{\mu\nu;1}(\alpha\beta;\ \xi\eta)=\sum\kappa_{\mu\nu}(\alpha\beta;\ ab,\ cd)\varepsilon(ab,\ cd;\ \xi\eta)$$

leads to an expression

$$\kappa_{\mu\nu;1}(\alpha\beta;\ \xi\eta)=\overline{A}_{\xi\eta}+L_{\mu\nu}Q(\alpha\beta;\ \xi\eta)+2^{-\lambda}T(\alpha\beta;\ \xi\eta),$$

where we put

$$L_{\mu\nu}=2^{-\mu}+2^{-\nu},\qquad \lambda=\mu+\nu-1.$$

The values of the quantity defined by

$$T(\alpha\beta;\ \xi\eta)=2\{\kappa_{11;1}(\alpha\beta;\ \xi\eta)-\kappa(\alpha\beta;\ \xi\eta)\}$$

are set out in the following lines:

$$T(ii;\ ii)=\tfrac{1}{4}(1-i)(2-i),\qquad T(ii;\ ik)=-\tfrac{1}{2}k(2-i),$$
$$T(ii;\ kk)=\tfrac{1}{4}k(1+k),\qquad T(ii;\ hk)=\tfrac{1}{2}hk;$$
$$T(ij;\ ii)=\tfrac{1}{8}(1-2i+2i^2),\qquad T(ij;\ ij)=\tfrac{1}{4}(1-2i-2j+2ij),$$
$$T(ij;\ ik)=-\tfrac{1}{2}k(1-i),\qquad T(ij;\ kk)=\tfrac{1}{4}k(1+k),$$
$$T(ij;\ hk)=\tfrac{1}{2}hk.$$

It can be shown that there hold the relations

$$\sum W(\alpha\beta;\, ab,\, cd)\varepsilon(ab,\, cd;\, \xi\eta) = Q(\alpha\beta;\, \xi\eta) + T(\alpha\beta;\, \xi\eta),$$
$$\sum T(\alpha\beta;\, ab,\, cd)\varepsilon(ab,\, cd;\, \xi\eta) = T(\alpha\beta;\, \xi\eta), \qquad \sum T(\alpha\beta;\, ab) = 0,$$
$$\sum \overline{A}_{ab} Q(\alpha\beta;\, cd)\varepsilon(ab,\, cd;\, \xi\eta) = \tfrac{1}{2} Q(\alpha\beta;\, \xi\eta).$$

2. Mother-descendant combination distant after a consanguineous marriage

The reduced probability in *generic case* with $n>1$ is defined by an equation
$$\kappa_{\mu\nu;n}(\alpha\beta;\, \xi\eta) = \sum \kappa_{\mu\nu;1}(\alpha\beta;\, ab)\kappa_{n-1}(ab;\, \xi\eta),$$
which is brought into the form
$$\kappa_{\mu\nu;n}(\alpha\beta,\, \xi\eta) = \overline{A}_{\xi\eta} + 2^{-n+1} L_{\mu\nu} Q(\alpha\beta;\, \xi\eta).$$
In fact, it is proved that there holds identically
$$\sum T(\alpha\beta;\, ab) Q(ab;\, \xi\eta) = 0.$$

Asymptotic behaviors of $\kappa_{\mu\nu;n}$ as μ, ν, or n tends to ∞ will be obvious. In fact, we obtain readily the limit equations
$$\lim_{\mu\to\infty} \kappa_{\mu\nu;n}(\alpha\beta;\, \xi\eta) = \kappa_{\nu+n}(\alpha\beta;\, \xi\eta), \qquad \lim_{\nu\to\infty} \kappa_{\mu\nu;n}(\alpha\beta;\, \xi\eta) = \kappa_{\mu+n}(\alpha\beta;\, \xi\eta),$$
and
$$\lim_{n\to\infty} \kappa_{\mu\nu;n}(\alpha\beta;\, \xi\eta) = \overline{A}_{\xi\eta},$$
among which first two remain valid also for $n=1$.

3. General mother-descendant combination through a single consanguineous marriage

In the present section we consider a general mother-descendant combination in which there concerns an intermediate collateral separation as well as a subsequent consanguineous marriage. Let namely an individual $A_{\alpha\beta}$ originate an lth descendant where a collateral separation takes place, and let the $(\mu,\, \nu)$th descendants of the latter be then married consanguineously and produce an nth descendant $A_{\xi\eta}$. Let the probability of combination $(A_{\alpha\beta};\, A_{\xi\eta})$ be then designated by
$$\pi_{l|\mu\nu;n}(\alpha\beta;\, \xi\eta) \equiv \overline{A}_{\alpha\beta}\kappa_{l|\mu\nu;n}(\alpha\beta;\, \xi\eta).$$
It is defined by an equation
$$\kappa_{l|\mu\nu;n}(\alpha\beta;\, \xi\eta) = \sum \kappa_l(\alpha\beta;\, ab)\kappa_{\mu\nu;n}(ab;\, \xi\eta).$$

The formula for the *lowest case* $n=1$ is exceptional and is expressed in the form
$$\kappa_{l|\mu\nu;1}(\alpha\beta;\, \xi\eta) = \overline{A}_{\xi\eta} + 2^{-l} L_{\mu\nu} Q(\alpha\beta;\, \xi\eta) + 2^{-\lambda} R(\xi\eta) + 2^{-l-\lambda} S(\alpha\beta;\, \xi\eta),$$
where we put, besides $\lambda = \mu + \nu - 1$,
$$R(\xi\eta) = \sum \overline{A}_{ab} T(ab;\, \xi\eta), \quad S(\alpha\beta;\, \xi\eta) = 2\sum Q(\alpha\beta;\, ab) T(ab;\, \xi\eta).$$

The values of these quantities are set out as follows:

$$R(ii)=\tfrac{1}{2}i(1-i),\qquad R(ij)=-ij;$$
$$S(ii;ii)=\tfrac{1}{4}(1-i)(1-2i),\qquad S(ii;ik)=-\tfrac{1}{2}k(1-2i),$$
$$S(ii;kk)=-\tfrac{1}{4}k(1-2k),\qquad S(ii;hk)=hk,$$
$$S(ij;ii)=\tfrac{1}{8}(1-2i)^2,\qquad S(ij;ij)=-\tfrac{1}{4}(i+j-4ij),$$
$$S(ij;ik)=-\tfrac{1}{4}k(1-4k).\qquad S(ij;kk)=-\tfrac{1}{4}k(1-2k),$$
$$S(ij;hk)=hk.$$

It would be noted that there hold the relations

$$\sum S(\alpha\beta;ab)=\sum \overline{A}_{ab}S(ab;\xi\eta)=0,$$
$$\sum \kappa(\alpha\beta;ab)S(ab;\xi\eta)=\sum Q(\alpha\beta;ab)S(ab;\xi\eta)=\tfrac{1}{2}S(\alpha\beta;\xi\eta).$$

The formula for *generic case* with $n>1$ is simply given by

$$\kappa_{l|\mu\nu;n}(\alpha\beta;\xi\eta)=\overline{A}_{\xi\eta}+2^{-l-n+1}L_{\mu\nu}Q(\alpha\beta;\xi\eta).$$

It is in passing noted that the following relations can be proved:

$$\sum R(ab)=\sum R(ab)Q(ab;\xi\eta)=\sum S(\alpha\beta;ab)Q(ab;\xi\eta)=0,$$
$$\sum U(\alpha\beta;ab,cd)\varepsilon(ab,cd;\xi\eta)=\tfrac{1}{2}Q(\alpha\beta;\xi\eta)+\tfrac{1}{4}S(\alpha\beta;\xi\eta),$$
$$\sum V(\alpha\beta;ab,cd)\varepsilon(ab,cd;\xi\eta)=Q(\alpha\beta;\xi\eta)+\tfrac{1}{2}S(\alpha\beta;\xi\eta),$$
$$\sum S(\alpha\beta;ab,cd)\varepsilon(ab,cd;\xi\eta)=\tfrac{1}{2}S(\alpha\beta;\xi\eta).$$

Asymptotic behaviors of $\kappa_{l|\mu\nu;n}$ as one among the generation-numbers involved tends to ∞ will be obvious. In fact, we readily obtain the limit equations

$$\lim_{l\to\infty}\kappa_{l|\mu\nu;n}(\alpha\beta;\xi\eta)=\lim_{n\to\infty}\kappa_{l|\mu\nu;n}(\alpha\beta;\xi\eta)=\overline{A}_{\xi\eta},$$
$$\lim_{\mu\to\infty}\kappa_{l|\mu\nu;n}(\alpha\beta;\xi\eta)=\kappa_{l+\nu+n}(\alpha\beta;\xi\eta),\quad \lim_{\nu\to\infty}\kappa_{l|\mu\nu;n}(\alpha\beta;\xi\eta)=\kappa_{l+\mu+n}(\alpha\beta;\xi\eta).$$

4. Contracting factor and equivalent generation-number

The present section is devoted *to explain a meaning of the quantity*

$$L_{\mu\nu}\equiv 2^{-\mu}+2^{-\nu}$$

introduced in § 1, from a view-point of genetics.

As shown in § 2, the probability $\kappa_{\mu\nu;n}(n>1)$ of mother-descendant combination distant after a consanguineous marriage is expressed in the form

$$\kappa_{\mu\nu;n}(\alpha\beta;\xi\eta)=\overline{A}_{\xi\eta}+2^{-n+1}L_{\mu\nu}Q(\alpha\beta;\xi\eta).$$

On the other hand, the probability κ_{n^*} of mother-descendant combination without any consanguineous marriage has been established, in I, § 1 in the form

$$\kappa_{n^*}(\alpha\beta;\xi\eta)=\overline{A}_{\xi\eta}+2^{-n^*+1}Q(\alpha\beta;\xi\eta).$$

The comparison of these formulas will well interpret a meaning of the factor $L_{\mu\nu}$. In fact, we introduce a positive number ρ by an equation

$$2^{-\rho}=L_{\mu\nu}\qquad(\rho\equiv\rho_{\mu\nu})$$

which is solved in the explicit form

$$\rho = -\log L_{\mu\nu}/\log 2 = \mu + \nu - \log(2^{\mu} + 2^{\nu})/\log 2.$$

The probability $\kappa_{\mu\nu;n}$ is then brought into the form

$$\kappa_{\mu\nu;n}(\alpha\beta;\,\xi\eta) = \overline{A}_{\xi\eta} + 2^{-(n+\rho)+1}Q(\alpha\beta;\,\xi\eta)$$

which coincides formally with $\kappa_{n+\rho}(\alpha\beta;\,\xi\eta)$, though the number ρ is, in general, i. e. unless $\mu=\nu$, not equal to an integer.

We can thus state the following proposition. *A consanguineous marriage between collateral* $(\mu,\ \nu)$*th descendants produces such an effect on consanguineous intimacy between an original and a further* $n(>1)$*th descendant originated from the consanguineous marriage that the part from the original individual to the* $(\mu,\ \nu)$*th descendants can be replaced by a lineal combination with a generation-number* ρ *defined as above which is equal to at most* $\mathrm{Max}(\mu,\ \nu)-1$ *and at least* $\mathrm{Min}(\mu,\ \nu)-1$.

It should be noted that the proposition does *not* remain valid for an exceptional case $n=1$.

By reason of its own meaning explained just above, we call the number $\rho_{\mu\nu}$ an *equivalent generation-number* and the factor $L_{\mu\nu}$ a *contracting factor*.

In conclusion, it would be noticed that for practical purpose of computing the values of ρ's and of L's it suffices to obtain the values of these quantities with one generation-number equal to 1. In fact, besides an evident symmetry character with respect to generation-numbers, they satisfy the recurrence equations

$$L_{\mu+1,\nu+1} = 2^{-1}L_{\mu\nu} \quad \text{and} \quad \rho_{\mu+1,\nu+1} = \rho_{\mu\nu} + 1,$$

which yield the desired relations

$$L_{\mu\nu} = 2^{-(\nu-1)}L_{\mu-\nu+1,1} \quad \text{and} \quad \rho_{\mu\nu} = \rho_{\mu-\nu+1,1} + \nu - 1$$

provided $\mu \geqq \nu \geqq 1$.