34. Note on Dirichlet Series. XII. On the Analogy between Singularities and Order-Directions. I

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(1) **Introduction.** Let us put

$$(1.1) \quad F(s)=\sum_{n=1}^{\infty} a_n \ \exp(-\lambda_n s) \quad (s=\sigma+it, \quad 0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n\to+\infty).$$

C. Biggeri has proved the next theorem.

C. Biggeri's Theorem (1) pp. 979–980, 2) p. 294). *Let (1.1) be simply convergent for* $\sigma>0$. *If* $\Re(a_n)\geqq 0$ $(n=1, 2,\ldots)$ *and* $\lim_{n\to+\infty}(\cos(\arg(a_n)))^{1/\lambda_n}=1$, *then* $s=0$ *is the singular point.*

In this note, we shall establish an analogous theorem concerning order-direction. We begin with

Definition. *Let (1.1) be uniformly convergent in the whole plane. Then, we call the direction* $\Im(s)=t$ *the order-direction of (1.1), provided that, in* $|\Im(s)-t|\leqq\varepsilon$ (ε : *any positive constant*), *(1.1) has the same order as in the whole plane, i.e.*

$$\overline{\lim_{\sigma\to-\infty}} 1/(-\sigma).\ \log^+\log^+ M(\sigma)=\overline{\lim_{\sigma\to-\infty}} 1/(-\sigma).\ \log^+\log^+ M(\sigma, t, \varepsilon),$$

where $M(\sigma)=\operatorname{Sup}_{-\infty<t<+\infty}|F(\sigma+it)|$, $\quad M(\sigma, t, \varepsilon)=\operatorname{Max}_{\Re(s)=\sigma, |\Im(s)-t|\leqq\varepsilon}|F(s)|$,

$\log^+ x = \operatorname{Max}\{0, \log x\}$.

Remark. The order-direction is a special case of the order-curve defined in the previous note (3)).

Our theorem is the following

Theorem. *Let (1.1) be uniformly convergent in the whole plane. If we have*

$$(1.2) \quad \begin{array}{ll} \text{(i)} & \Re(a_n)\geqq 0 \qquad (n=1, 2,\ldots), \\ \text{(ii)} & \lim_{n\to\infty} 1/\lambda_n \log\lambda_n \cdot\log(\cos\theta_n)=0, \quad \arg(a_n)=\theta_n, \end{array}$$

then $\Im(s)=0$ *is the order-direction of (1.1).*

As its corollary, we get

Corollary. *Let (1.1) with* $\Re(a_n)\geqq 0$ $(n=1, 2,\ldots)$, $\lim_{n\to\infty}(\cos\theta_n)^{1/\lambda_n}=1$, $(\theta_n=\arg(a_n))$ *be simply (necessarily absolutely) convergent in the whole plane. Then* $\Im(s)=0$ *is the order-direction of (1.1). In particular, if* $|\theta_n|\leqq\theta<\pi/2$ $(n=1,2,\ldots)$, *the same conclusion holds.*

(2) **Lemmas.** To prove this theorem, we need some lemmas.

Lemma I (C. Tanaka, 4) p. 77, corollary IV). *Under the same assumptions as in our Theorem, the order* ρ *of (1.1) is given by*

$$(2.1)\qquad -1/\rho=\overline{\lim_{x\to\infty}}\,(x\log x)^{-1}.\log T_x,$$

where (i) $\rho=\overline{\lim_{\sigma\to-\infty}}\,1/(-\sigma).\log^+\log^+ M(\sigma),\quad M(\sigma)=\sup_{-\infty<t<+\infty}|F(\sigma+it)|,$

(ii) $T_x=\sup_{-\infty<t<+\infty}|\sum_{[x]\leqq\lambda_n<x} a_n\exp(-it\lambda_n)|,\quad [x]$: *the greatest integer contained in* x.

Lemma II. *Let (1.1) with* $\Re(a_n)\geqq 0$ $(n=1,2,\dots)$ *be uniformly convergent in the whole plane. Put* $G(s)=\sum_{n=1}^{\infty}\Re(a_n)\exp(-\lambda_n s)$, *which is evidently absolutely convergent everywhere. If* $G(s)$ *has the same order as (1.1), then* $\Im(s)=0$ *is the order-direction of (1.1).*

Proof. We denote by ρ and ρ_G the order of (1.1) and $G(s)$ respectively. For any given ε (>0), we have easily

$$\sup_{-\infty<t<+\infty}|F(\sigma+it)|=M(\sigma)\geqq\max_{\Re(s)=\sigma,\ |\Im(s)|\leqq\varepsilon}|F(s)|=M(\sigma;\varepsilon)\geqq|F(\sigma)|$$

$$\geqq\left|\Re F(\sigma)\right|=\sum_{n=1}^{\infty}\Re(a_n)\exp(-\lambda_n\sigma)=\sup_{-\infty<t<+\infty}|G(\sigma+it)|=M_G(\sigma),$$

so that

$$\rho=\overline{\lim_{\sigma\to-\infty}}\,1/(-\sigma).\log^+\log^+M(\sigma)\geqq\overline{\lim_{\sigma\to-\infty}}\,1/(-\sigma).\log^+\log^+M(\sigma;\varepsilon)$$

$$\geqq\overline{\lim_{\sigma\to-\infty}}\,1/(-\sigma).\log^+\log^+M_G(\sigma)=\rho_G.$$

Hence, by $\rho=\rho_G$, we get

$$\rho=\overline{\lim_{\sigma\to-\infty}}\,1/(-\sigma).\log^+\log^+M(\sigma;\varepsilon).$$

Since ε is arbitrary, $\Im(s)=0$ is the order-direction of (1.1).

(3) **Proof of the Theorem**

By lemma II, it suffices to prove that $G(s)=\sum_{n=1}^{\infty}\Re(a_n)\exp(-\lambda_n s)$ has the same order as $F(s)$. Denoting by ρ, ρ_G, the order of $F(s)$ and $G(s)$ respectively, by lemma I we have

$$\overline{\lim_{x\to\infty}}\,1/(x\log x).\log T_x=-1/\rho,\quad T_x=\sup_{-\infty<t<+\infty}|\sum_{[x]\leqq\lambda_n<x}a_n\exp(-it\lambda_n)|,$$

$$\overline{\lim_{x\to\infty}}\,1/(x\log x).\log U_x=-1/\rho_G,\quad U_x=\sup_{-\infty<t<+\infty}|\sum_{[x]\leqq\lambda_n<x}\Re(a_n)\exp(-it\lambda_n)|$$

$$=\sum_{[x]\leqq\lambda_n<x}\Re(a_n)\qquad(\Re(a_n)\geqq 0).$$

Since $T_x\geqq|\sum_{[x]\leqq\lambda_n<x}a_n|\geqq\sum_{[x]\leqq\lambda_n<x}\Re(a_n)=U_x$, we have evidently

$$(3.1)\qquad -1/\rho\quad\geqq\quad -1/\rho_G.$$

On the other hand, putting $\min_{[x]\leqq\lambda_n<x}\cos(\theta_n)=\cos\theta_{n(x)}$, we get

$$(3.2)\qquad U_x=\sum_{[x]\leqq\lambda_n<x}\Re(a_n)=\sum_{[x]\leqq\lambda_n<x}|a_n|\cos(\theta_n)\geqq\cos(\theta_{n(x)}).\sum_{[x]\leqq\lambda_n<x}|a_n|$$

$$\geqq\cos(\theta_{n(x)}).T_x.$$

Since we get easily

$$\overline{\lim_{x\to\infty}}\,1/(x\log x).\,\lambda_{n(x)}\log\lambda_{n(x)}=1,\quad \lim_{x\to\infty}1/(\lambda_{n(x)}\log\lambda_{n(x)}).\log\cos(\theta_{n(x)})=0,$$

we obtain

$$\lim_{x\to\infty} 1/(x\log x)\,.\log(\cos\theta_{n(x)})=0,$$

so that, by (3.2)

$$\overline{\lim_{x\to\infty}}\, 1/(x\log x)\,.\log U_x \geqq \overline{\lim_{x\to\infty}}\, 1/(x\log x)\,.\log T_x\,,\ \text{i.e.}$$

(3.3) $$-1/\rho_G \quad \geqq \quad -1/\rho.$$

Hence, by (3.1), (3.2), $\rho=\rho_G$. q.e.d.

Proof of Corollary. By T. Kojima's theorem (5)), the simple and absolute convergence-abscissa σ_s, σ_a of (1.1) are determined respectively by

$$\sigma_s=\overline{\lim_{x\to\infty}}\, 1/x.\log\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big| = -\infty\ ,\quad \sigma_a=\overline{\lim_{x\to\infty}}\, 1/x\,.\log\Big\{\sum_{[x]\leqq\lambda_n<x}|a_n|\Big\}.$$

Therefore, putting $\underset{[x]\leqq\lambda_n<x}{\text{Min}}\cos(\theta_n)=\cos(\theta_{n(x)})$, we have

$$\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big| \geqq \sum_{[x]\leqq\lambda_n<x}\Re(a_n) = \sum_{[x]\leqq\lambda_n<x}|a_n|\cos(\theta_n) \geqq \cos(\theta_{n(x)})\,.\sum_{[x]\leqq\lambda_n<x}|a_n|,$$

so that

(3.4) $$-\infty=\sigma_s=\overline{\lim_{x\to\infty}}\, 1/x.\log\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big|$$

$$\geqq \overline{\lim_{x\to\infty}}\, 1/x.\log\Big\{\sum_{[x]\leqq\lambda_n<x}|a_n|\Big\} + \lim_{x\to\infty} 1/\lambda_{n(x)}.\log(\cos(\theta_{n(x)}))\,.\lambda_{n(x)}/x.$$

Since we get evidently

$$\lim_{x\to\infty}\lambda_{n(x)}/x=1,\qquad \lim_{x\to\infty} 1/\lambda_{n(x)}.\log(\cos(\theta_{n(x)}))=0,$$

by (3.4) we obtain $\sigma_a=-\infty$. In other words, $F(s)$ is absolutely (a fortiori uniformly) convergent in the whole plane. Thus, all assumptions of theorem are satisfied, so that $\Im(s)=0$ is the order-direction, q.e.d.

References

1) C. Biggeri: Sur les singularités des fonctions analytiques définies par des séries de Dirichlet. C. R. Acad. Sci. Paris, **209** (1939).

2) C. Tanaka: Note on Dirichlet series (II). On the singularities of Dirichlet series (II). Tôhoku Math. Journ., **3**, No. 3 (1951).

3) ——: Note on Dirichet series. XI. On the analogy between singularities and order-curves. Proc. Japan Acad., **29**, No. 9 (1953).

4) ——: Note on Dirichlet series (V). On the integral functions defined by Dirichlet series (I). Tôhoku Math. Journ., **5**, No. 1 (1953).

5) T. Kojima: On the convergence-abscissa of general Dirichlet series, Tôhoku Math. Journ., **6** (1914–'15).