63. On Locally Convex Vector Spaces of Continuous Functions

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1. In the present note we study locally convex spaces, with the compact-open topology, of real valued continuous functions on completely regular spaces. Furthermore using its results we give an answer to a problem proposed by N. Bourbaki and J. A. Dieudonné:[1] Does there exist a t-space which is not bornologic?

Concerning vector spaces we adopt the notation of J. A. Dieudonné[1] and consider only vector spaces over the real field.

2. Let X be a completely regular space and let $\mathfrak{C}(X, R)$ be a locally convex vector space, with the compact-open topology, of all real valued continuous functions on X. Then we prove the following

Theorem 1. *In order that the $\mathfrak{C}(X, R)$ is a t-space, it is necessary and sufficient that X satisfies the following condition:*

(Q_1) any closed and relatively precompact[2] *subset of X is compact.*

To prove this we shall need the following lemmas.

Lemma 1. *Let F be a non-zero continuous linear function on $\mathfrak{C}(X, R)$. Then there is the minimal compact non-void subset K of X such that $D(f)\cap K=\phi$ implies $F(f)=0$, where $D(f)=\{x|x \in X \ \& \ f(x)\neq 0\}$.*

Proof. Let $\mathfrak{F}$ be the family of all compact subsets C such that if $D(f)\cap C=\phi$ then $F(f)=0$. Since F is continuous, $\mathfrak{F}$ is not void. Moreover F is non-zero, hence $\mathfrak{F}$ satisfies the finite intersection property and in fact $\mathfrak{F}$ is an ideal, i.e., if F_1 and F_2 belong to $\mathfrak{F}$ then $F_1 \cap F_2 \in \mathfrak{F}$. Accordingly the intersection K of all subsets of $\mathfrak{F}$ is non-void and belongs to $\mathfrak{F}$. Thus we see that K is the required one.

From now on the set K in Lemma 1 will be called the *carrier* of F and denoted by K_F (for the zero function 0 let $K_0=\Phi$).

Lemma 2. *Let B' be a weakly bounded subset of $\mathfrak{C}(X, R)'$. Then the closure C of the sum of all carriers of F in B' is relatively precompact.*

Proof. Suppose that the lemma is not true. Then there are a function f of $\mathfrak{C}(X, R)$ and a countable subset $\{x_n\}$ of C such that $|f(x_n)|\to\infty$ as $n\to\infty$. Then we may assume without loss of generality that any $x_n \in K_{F_n}$ for some $F_n \in B'$ and that $f(x_{n+1})>f(x_n)+1$ for $n=1, 2, \dots$. Let U_n be a neighbourhood of x_n such that $U_n =\{x\,\big|\,|f(x)-f(x_n)|<1/2\}$. Then for $n\neq m$ $U_n\cap U_m=\phi$ and for any sequence $\{f_n \mid f_n \in \mathfrak{C}(X, R) \ \&\ D(f_n) \subset U_n\}$, $\sum_{n=1}^{\infty} f_n \in \mathfrak{C}(X, R)$. Now we assert that there is a sequence $\{f_{n_i} \mid i=1, 2, \dots\}$ such that $F_{n_i}(f_{n_1} + f_{n_2} + \cdots + f_{n_i}) = n_i$ and $D(f_{n_i}) \cap K_{F_{n_j}} = \phi$ for $i > j$ and such that $D(f_{n_i}) \subset U_{n_i}$. To prove this assume that $\{f_{n_j} \mid j < i\}$ has been already constructed, and find U_{n_i} such that $\sum_{j<i} K_{F_{n_j}} \cap U_{n_i} = \phi$, which is possible, since the carrier of F is compact by Lemma 1. Then we show that there is an $f \in \mathfrak{C}(X, R)$ such that $D(f) \subset U_{n_i}$ and $F_{n_i}(f) \neq 0$. For suppose $D(f) \subset U_{n_i}$ implies $F_{n_i}(f) = 0$. Let U' be a neighbourhood of x_{n_i} such that $\overline{U'} \subset U_{n_i}$ and such that $\rho(X - U_{n_i}) \equiv 1$ and $\rho(\overline{U'}) \equiv 0$ for some $\rho \in \mathfrak{C}(X, R)$. Then for any $f \in \mathfrak{C}(X, R)$ such that $D(f) \cap (K_{F_{n_i}} - U') = \phi$, $f = \rho f + (1-\rho) f$, $D(\rho f) \cap K_{F_{n_i}} = \phi$ and $D((1-\rho) f) \subset U_{n_i}$ hence $F_{n_i}(f) = 0$. This implies that $K_{F_{n_i}} - U'$ contains the carrier $K_{F_{n_i}}$ of F_{n_i}, which is a contradiction. Hence there exists an f such that $D(f) \subset U_{n_i}$ and $F_{n_i}(f) \neq 0$. Then for some k $F_{n_j}(f_{n_1} + \cdots + f_{n_{i-1}} + kf) = n_i$. Thus by setting $kf = f_{n_i}$, we obtain the sequence in question by induction.

Now let $f = \sum_{i=1}^{\infty} f_{n_i}$. Then $f \in \mathfrak{C}(X, R)$ and for any i $F_{n_i}(f) = F_{n_i}(\sum_{j=1}^{i} f_{n_j}) + F_{n_i}(\sum_{j>i} f_{n_j}) = F_{n_i}(\sum_{j=1}^{i} f_{n_j}) = n_i$, which implies that B' is not weakly bounded.

The proof of Theorem 1. Let $\mathfrak{C}(X, R)$ be a t-space and let C be a closed and relatively precompact subset of X. Then $B' = \{F_x \mid x \in C\}$ is weakly bounded, where $F_x(f) = f(x)$. But if C is not compact, there is a maximal filter $\mathfrak{F}$ of C such that it has no limits in C. Accordingly we can find a point ∞ of $\beta(C) - C$ such that $\mathfrak{F}$ converges to ∞, where $\beta(C)$ is the Čech-compactification of C. Then $\{\{F_x \mid x \in G\} \mid G \in \mathfrak{F}\}$ weakly converges to F_∞ where $F_\infty(f)$ is the value of the extension, at ∞, of f over $\beta(C)$. However, F_∞ has no carrier in X. Hence by Lemma 1 F_∞ is not continuous. This means that B' is not weakly relatively compact, which is a contradiction.[3] Accordingly C is compact and so X satisfies the condition (Q_1).

Conversely, let X be a space satisfying the condition (Q_1). Then we have only to prove that any weakly bounded subset B' of $\mathfrak{C}(X, R)'$ is equi-continuous. Now by Lemma 2 and by our assumption for any weakly bounded subset B' there is a compact subset K of X such that any $F \in B'$ has the carrier contained in K. Let $\sigma(F)$ be an element of $\mathfrak{C}(K, R)'$ such that for any $f \in \mathfrak{C}(K, R)$ $\sigma(F)(f) = F(\bar{f})$ where $\bar{f}$ is an extension of f over X. Then $\{\sigma(F) \mid F \in B'\}$ is a weakly bounded subset of $\mathfrak{C}(K, R)'$ and $\mathfrak{C}(K, R)$ is a Banach space, hence it is a t-space. Therefore we can find an $\varepsilon > 0$ such that $f \in \mathfrak{C}(K, R)$ and $\|f\| \leqq \varepsilon$ implies $|F(f)| \leqq 1$ for any $F \in \sigma(B')$.

Hence $f \in \mathfrak{C}(X, R)$ and $|f(x)| \leqq \varepsilon$ on K implies $|F(f)| \leqq 1$ for any $F \in B'$, which shows that B' is equi-continuous.

3. In general a locally convex vector space is bornologic if and only if it is a boundedly closed and quasi-t-space. However in our special case we obtain the following theorems.

Theorem 2. *Under the same assumption as in Theorem 1, the following conditions on X are equivalent:*

(1) *X is a Q-space,*[4]

(2) *$\mathfrak{C}(X, R)$ is boundedly closed,*[5]

(3) *$\mathfrak{C}(X, R)$ is bornologic.*

Proof. We first show that (1) implies (3). Let X be a Q-space. Then we see that X satisfies the condition (Q_1) of Theorem 1, hence $\mathfrak{C}(X, R)$ is a t-space and so is a quasi-t-space. Furthermore let F be a real valued linear function on $\mathfrak{C}(X, R)$, which transforms all bounded subsets into bounded subsets. Then F is a bounded functional in the sense of lattice $\mathfrak{C}(X, R)$, i.e., for any f there is a positive number r_f such that $|g| \leqq |f|$ implies $|F(g)| \leq r_f$. Hence[6] F is continuous, which implies that $\mathfrak{C}(X, R)$ is bornologic.

Obviously (3) implies (2).

To prove that (2) implies (1), suppose that X is not a Q-space. Then there exists a point ∞ of $e(X)-X$. Now let F_∞ be a linear function on $\mathfrak{C}(X, R)$ such that for any $f \in \mathfrak{C}(X, R)$ $F_\infty(f) = \bar{f}(\infty)$, where f is an extension of f over $e(X)$. Then F_∞ transforms any bounded subset into bounded subset. For if $F_\infty(B)$ is not bounded for some bounded subset B of $\mathfrak{C}(X, R)$, then there is a sequence $\{f_n | f_n \in B\}$ such that $F_\infty(f_n) = r_n$ and $|r_n| \to \infty$, as $n \to \infty$. Then we show that $\overset{\infty}{\underset{n=1}{U}} D(f_n - r_n) \neq X$. For if $\overset{\infty}{\underset{n=1}{U}} D(f_n - r_n) = X$ $f = \sum_{n=1}^\infty (1/2^n \wedge |f_n - r_n|)$ is strictly positive, hence its extension over $e(X) f = \sum_{n=1}^\infty (1/2^n \wedge |\bar{f}_n - r_n|)$ is strictly positive. But $f_n(\infty) = r_n$ implies $f(\infty) = 0$, which is a contradiction. Thus we see that $f_n(x) = r_n$ for some point x of X, which also contradicts the boundedness of B. On the other hand F_∞ has no carrier, hence by Lemma 1 it is not continuous, which means that $\mathfrak{C}(X, R)$ is not boundedly closed.

Theorem 3. *Let X be a locally compact space and let $\mathfrak{C}_k(X, R)$ be a locally convex vector space, with the compact-open topology, of all real valued continuous functions on X which have compact carriers. Then the following conditions on X are equivalent:*

(1) *X satisfies the condition (Q_1) in Theorem 1,*

(2) *$\mathfrak{C}_k(X, R)$ is a quasi-t-space,*

(3) *$\mathfrak{C}_k(X, R)$ is bornologic.*

Proof. We first show that (2) implies (1). Let C be a closed and relatively precompact subset of X and let $V = \{f | f \in \mathfrak{C}_k(X, R)$

& $|f(x)|\leqq 1$ for any $x \in C\}$. Then V is a barrel and absorbs every bounded set, since any bounded set of $\mathfrak{C}_k(X, R)$ is uniformly bounded on C. Hence if $\mathfrak{C}_k(X, R)$ is a quasi-t-space, V is a neighbourhood of 0 in $\mathfrak{C}_k(X, R)$. Accordingly for some compact subset K of X and for some positive number r, $|f(x)|\leqq r$ on K implies $f \in V$, from which follows that C is a subset of K. Hence L is compact.

Obviously (3) implies (2).

We finally show that (1) implies (3). Let V be a convex, symmetric subsets absorbing every bounded subset of $\mathfrak{C}_k(X, R)$ and let C be the sum of all $D(f)$ such $f \notin Y$. Then we show that C is relatively precompact. For if this is not true, we can find a family $\{D(f_n)\}$ such that $D(f_n)\cap D(f_m)=\phi$ for $n\neq m$ and any compact subset meets only a finite number of members of $\{D(f_n)\}$. Then $\{nf_n\}$ is a bounded subset of $\mathfrak{C}_k(X, R)$ and $nV \not\ni nf$, which is a contradiction. Hence C is relatively precompact. Accordingly if X satisfies (Q_1), $\bar{C}$ is compact. Now let $B=\{f \mid D(f)\subset V(\bar{C})$ & $\|f\|\leqq 1\}$ where $U(\bar{C})$ is a compact neighbourhood of C. Then B is bounded, hence for some $\lambda>0$, $B\subset\lambda V$. Therefore if f is a function in $\mathfrak{C}_k(X, R)$ such that $|f(x)|\leqq 1/2\lambda$ for any $x \in U(\bar{C})$, $f \in V$, which implies that V is a neighbourhood of 0 in $\mathfrak{C}_k(X, R)$.

4. **Example.** Let $W(\omega_2)$ be the space, of all ordinals less than the initial ordinal ω_2 of the fourth class, with the interval topology and let L be the subspace of $W(\omega_2)$ whose elements are not ω_0-limits. Then the space L is $\aleph_1$-additive[7] and is not a Q-space since it has no complete structures. Furthermore by the normality and the $\aleph_1$-additivity of L any closed and relatively precompact subset of L is finite, hence L satisfies the condition (Q_1). Thus by Theorems 1 and 2 $\mathfrak{C}(L, R)$ with the compact-open topology is just an example of the space which is a t-space but is not bornologic.

We finally remark that if X has a complete structure then it satisfies the condition (Q_1) and that for any space X the condition (Q_1) implies the following condition: (Q_2) any closed and countable compact subset of X is compact, but that in general the converse is not true. For in the product space T of the spaces $W(\omega_1+1)$ and $W(\omega_0+1)$ with the interval topologies respectively let T_1 be the subspace of T: $T-\{(\alpha, \omega_0)|\alpha$ is a limit point in $W(\omega_1+1)\}$.

Then the resulting space T_1 satisfies the condition (Q_2) but does not enjoy the condition (Q_1).

References

1) Cf. N. Bourbaki: Sur certains spaces vectoriels topologiques, Ann. Inst. Fourier II (1950). J. A. Dieudonné: Recent developments in the theory of locally convex spaces, Bull. Amer. Math. Soc., **59** (1953).

2) We say that a subset Y of topological space X is *relatively precompact* if any real valued continuous function on X is bounded on Y.

3) L.c. 1).

4) Cf. E. Hewitt: Rings of real valued continuous functions, Trans. Amer. Math. Soc., **64** (1948). T. Shirota: A class of topological spaces, Osaka Math. J., **4** (1952). L. Gillman and H. Henriksen: Concerning rings of continuous functions, Trans. Amer. Math. Soc. (to appear).

5) Cf. G. Mackey: On infinite-dimensional linear spaces, Trans. Amer. Math. Soc., **57** (1945). W. F. Donoghue and K. T. Smith: On the symmetry and bounded closure of locally convex spaces, Trans. Amer. Math. Soc., **73** (1952).

6) Cf. E. Hewitt: Linear functionals on spaces of continuous functions, Fund. Math., **37** (1950). L. Nachbin: On the continuity of positive linear transformations, proceeding of the international congress of mathematicians (1950).

7) Cf. R. Sikorski: Remark on some topological spaces of high power, Fund. Math., **37** (1950).