55. A Note on the Structure of Commutative Semigroups

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The object of the present note is to develop the structure theory of commutative semigroups. By a semigroup we shall always mean a commutative semigroup with identity element 1 and zero element 0.[1] If semigroup S has no identity and zero elements, it can always be imbedded in another S', which has them. S' consists of the elements of S together with new elements 1 and 0. The product of two elements $x, y \in S'$ is defined to be the old product xy of S if $x, y \in S$, otherwise $x0=0=0x$ and $x1=x=1x$ for all $x \in S'$. Moreover, every ideal[2] of S is again an ideal of S' and every principal ideal[3] of S which is generated by an element $x \in S$ is also a principal ideal of S' generated by the same element. Therefore, the assumption that a semigroup has identity and zero elements does not restrict us.

Let S be a semigroup (we recall our convention that "semigroup" means a commutative semigroup with identity element and zero element) and p an element of S, and we define the following (p)-equivalence relation in S:

Two elements a and b of S are *(p)-equivalent* (denoted by $a\overset{p}{\sim}b$) if and only if

$$\bigcap_{n=1}^{\infty}(Sp\cdot a^n)=\bigcap_{n=1}^{\infty}(Sp\cdot b^n).$$

Then it is clear that the (p)-equivalence relation satisfies the following equivalence relations:

$$(1') \qquad a\overset{p}{\sim}a \quad \text{for all} \quad a \in S,$$

$$(2') \qquad \text{if} \quad a\overset{p}{\sim}b \quad \text{then} \quad b\overset{p}{\sim}a,$$

$$(3') \qquad \text{if} \quad a\overset{p}{\sim}b \quad \text{and} \quad b\overset{p}{\sim}c \quad \text{then} \quad a\overset{p}{\sim}c.$$

Now we define the new equivalence relation (denoted by $\sim$), using the above (p)-equivalence relation, in S as follows:

$$a\sim b \quad \textit{if and only if} \quad a\overset{p}{\sim}b \quad \textit{for all} \quad p \in S.$$

It is easy to see that the relation $\sim$ satisfies the following equivalence relations:

$$(1) \qquad a\sim a \quad \text{for all} \quad a \in S,$$

$$(2) \qquad \text{if} \quad a\sim b \quad \text{then} \quad b\sim a,$$

$$(3) \qquad \text{if} \quad a\sim b \quad \text{and} \quad b\sim c \quad \text{then} \quad a\sim c.$$

In the discussion below, we denote by S_x the set of all elements in

S which are equivalent to x under the relation $\sim$. Clearly, either $S_x = S_y$ or $S_x \cap S_y = \phi$.

Lemmn 1. *The equivalence relation $\sim$ is multiplicatively invariant, that is, if $a \sim b$ then $xa \sim xb$ for any $x \in S$.*

Proof. Let p be an arbitrary element of S and j a positive integer, then from $a \sim b$ we have

$$\bigcap_{n=1}^{\infty}(Spx^j \cdot a^n) = \bigcap_{n=1}^{\infty}(Spx^j \cdot b^n).$$

Hence

$$\begin{aligned}\bigcap_{n=1}^{\infty}(Sp(xa)^n) &= \bigcap_{n=1}^{\infty}(Spx^n \cdot a^n) \\ &= \bigcap_{j=1}^{\infty}(\bigcap_{n=1}^{\infty}(Spx^j \cdot a^n)) = \bigcap_{j=1}^{\infty}(\bigcap_{n=1}^{\infty}(Spx^j \cdot b^n)) \\ &= \bigcap_{n=1}^{\infty}(Spx^n \cdot b^n) = \bigcap_{n=1}^{\infty}(Sp(xb)^n).\end{aligned}$$

Therefore $xa \overset{p}{\sim} xb$ and, as p is arbitrary, we have $xa \sim xb$.

Lemma 2. *Every S_x is a sub-semigroup of S.*

Proof. First, we show $a \sim a^2$ for all $a \in S$. Let p be an arbitrary element of S, then it is clear that

$$\bigcap_{n=1}^{\infty}(Sp \cdot a^n) = \bigcap_{n=1}^{\infty}(Sp \cdot (a^2)^n).$$

This shows that $a \overset{p}{\sim} a^2$. As p is arbitrary, $a \sim a^2$.

Now, let a, b be any two elements of S_x. Then $a \sim x \sim b$ and by Lemma 1 $ab \sim b^2 \sim b \sim x$. This implies $ab \in S_x$ and so S_x is a sub-semigroup of S.

Lemma 3. $S_x \cdot S_y \subset S_{xy}$.

Proof. If $a \in S_x$ and $b \in S_y$ then $a \sim x$, $b \sim y$. Hence by Lemma 1 $ab \sim ay \sim xy$ and $ab \in S_{xy}$, whence $S_x \cdot S_y \subset S_{xy}$.

Lemma 4. *If for idempotents e, f in S $e \sim f$, then $e = f$.*

Proof. If e, f are two idempotents $e \sim f$, then

$$Se = \bigcap_{n=1}^{\infty} Se^n = \bigcap_{n=1}^{\infty} Sf^n = Sf.$$

Hence $e = ef = f$.

Corollary. Every S_x has at most one idempotent.

Lemma 5. *If S_x contains an idempotent e then $S_x e$ is the group ideal (Suschkewitsch kernel[4]) of S_x.*

Proof. If ae, be are contained in $S_x e$ ($a, b \in S_x$) then

$$ae \cdot be = abe^2 = abe = (ab)e \in S_x e,$$

that is, $S_x e$ is a semigroup with identity e.

Let $a \in S_x$ then $a \sim e$ and $\bigcap_{n=1}^{\infty}(Se \cdot a^n) = \bigcap_{n=1}^{\infty}(Se \cdot e^n) = Se$. Therefore $Sea = Se$. Hence there exists an element a' in Se such that $a'a = e$ and $a'e = a'$. This implies $Sa' = Se$ and $Sa'^n = Se$ for $n = 1, 2, \ldots$. It follows $\bigcap_{n=1}^{\infty}(Sp \cdot a'^n) = Spe = \bigcap_{n=1}^{\infty}(Sp \cdot e^n)$ for any $p \in S$. Thus $a' \sim e$ and $a' \in S_x e$. This shows that an element ae has an inverse a' in $S_x e$, and $S_x e$ is a group.

Lemma 6. *The set-theoretical join $\bar{S}$ of those S_x's containing an idempotent is a sub-semigroup of S.*

Proof. Let a, b be any two elements of $\bar{S}$ then there exist idempotents e, f such that $a\sim e$, $b\sim f$. Hence, by Lemma 1, $ab\sim ef$ and ef is an idempotent. Thus $ab \in S_{ef}\subset\bar{S}$.

Henceforth, we denote by G_x the group ideal $S_x e$ of S_x containing an idempotent e. Then it is clear that if e and f are identities of groups G_x and G_y, respectively, then ef is the identity of the group G_{xy}.

Lemma 7. $G_x\cdot G_y\subset G_{xy}$.

Proof. From Lemma 3 $G_x\cdot G_y\subset S_{xy}$. If e and f are identities of groups G_x, G_y, respectively, then $G_{xy}=S_{xy}ef\supset G_xG_yef=G_xe\cdot G_yf=G_x\cdot G_y$.

From Lemmas 1–7 we have:

Theorem. *A commutative semigroup S is decomposed into sub-semigroups in the following way:*

$$S=(\bigcup S_x^*)\cup(\bigcup S_x),$$

where

(i) *S_x^*'s and S_x's are sub-semigroups of S having no element in common,*

(ii) *each semigroup S_x^* has no idempotent,*

(iii) *each semigroup S_x has one and only one idempotent e_x and $S_xe_x=G_x$ is the group ideal* (*Suschkewitsch kernel*) *of S_x,*

(iv) *the set theoretical join $\bar{S}$ of all S_x's having a group ideal is a sub-semigroup of S,*

(v) *$S_u\cdot S_v\subset S_{uv}$ (S_u or S_v may be a S_x^* or a S_x)*

(vi) *for group ideals $G_x\cdot G_y\subset G_{xy}$.*

Corollary 1. *If, in S, every element a satisfies the condition $Sa^n=Sa^{n+1}=Sa^{n+2}=\cdots$ for some positive integer n* (*n depending on a*) *then $S=\bar{S}$.*

Corollary 1.1. *If S satisfies the descending chain condition for ideals* (*or principal ideals*) *then $S=\bar{S}$.*

Corollary 1.2. *If, in S, every element is of finite order*[5] *then $S=\bar{S}$.*

Corollary 2. *If S is regular in the sense of J. v. Neumann*[6] *or, equivalently for the commutative case, S admits relative inverses,*[7] *then $S=\bigcup G_x$, where G_x's are groups satisfying the condition* (vi) *of the Theorem.*

After my investigation had been completed, Mr. T. Tamura at Tokushima University communicated to me that he had also obtained a similar result using a method different from mine.

References

1) A *semigroup* is a non-empty set closed to a single associative, binary multiplication. An *identity element* 1 of a semigroup S is an element of S with the property $1x=x=x1$ for all $x \in S$, a *zero element* 0 is an element of S such that $0x=0=x0$. Then it is clear that identity and zero is uniquely defined.

2) An *ideal* A of a commutative semigroup S is a non-empty subset of S with the property $SA \subset A$, where SA is the set of all elements sa, $s \in S$, $a \in A$.

3) A *principal ideal* P of S is an ideal generated by a single element, for example a principal ideal generated by an element x is equal to $Sx \cup x$ if S contains no identity, $=Sx$ if S contains identity.

4) Cf. Suschkewitsch: Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit, Math. Ann., **99**, 30–50 (1928).

5) An element a of a semigroup is said to be *finite order* if $a^r=a^t$ for some positive integers $r \neq t$. Cf. D. Rees: On semigroups, Proc. of Camb. Phil. Soc., **36**, 387–400 (1940).

6) An element a of a semigroup is called *regular* if and only if there exists $x \in S$ so that $axa=a$. If every element of S is regular then S is called a *regular semigroup*. This concept was introduced by J. v. Neumann for rings. Cf. J. v. Neumann: On regular rings, Proc. Nat. Acad. Sci. U.S.A., **22**, 707 (1936).

7) Cf. A. H. Clifford: Semigroups admitting relative inverses, Ann. Math., (2) **42**, 1037–1049.