76. Transgression and the Invariant k_n^{q+1}

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§ 1. Let X be a topological space with vanishing homotopy groups $\pi_i(X)$ for $i \neq n, q (1<n<q)$, and let $x_0 \in X$ be a base point. For the sake of brevity, we write in the following $\pi_n=\pi_n(X)$ and $\pi_q=\pi_q(X)$. We call a space of type (π, r) any space Y such that $\pi_i(Y)=0 (i \neq r)$ and $\pi_r(Y) \approx \pi$. Then, following Cartan-Serre,[1] we have the fiber space (E, p, B) in the sense of Serre[2] such that

i) the total space E is of the same homotopy type as X,

ii) the base space B is a space of type (π_n, n), and $X \subset B$,

iii) the fiber $F=p^{-1}(x_0)$ is a space of type (π_q, q).

Consider in this fiber space the transgression τ: $E^{*0,q}_{q+1} \xrightarrow{d^{q+1}} E^{*q+1,0}_{q+1}$ of the singular cohomology spectral sequence with coefficients in π_q.[2] Then, since the singular homology group $H_i(F; \pi_q)=0$ for $i<q$, we have $E^{*0,q}_{q+1}=H^q(F; \pi_q)$, $E^{*q+1,0}_{q+1}=H^{q+1}(B; \pi_q)$ and

$$\tau=p^{*-1}\circ\delta^* : H^q(F; \pi_q) \longrightarrow H^{q+1}(B; \pi_q),$$

where $\delta^* : H^q(F; \pi_q) \longrightarrow H^{q+1}(E, F; \pi_q)$ is the coboundary operator, and $p^* : H^{q+1}(B; \pi_q) \longrightarrow H^{q+1}(E, F; \pi_q)$ is the homomorphism induced by p. Let $\boldsymbol{b}^q \in H^q(F; \pi_q)$ be the basic cohomology class,[3] and let $\overline{\boldsymbol{k}}_n^{q+1} \in H^{q+1}(B; \pi_q)$ be the geometrical realization of the Eilenberg-MacLane invariant $\boldsymbol{k}_n^{q+1} \in H^{q+1}(\pi_n, n; \pi_q)$ of the space X.[4] Then $\boldsymbol{b}^q$ and $\overline{\boldsymbol{k}}_n^{q+1}$ are related by τ as follows:

$$\tau\boldsymbol{b}^q = -\overline{\boldsymbol{k}}_n^{q+1}. \tag{1.1}$$

The main purpose of the present note is to give a proof of (1.1). The proof is given by making use of the theory of J. H. C. Whitehead.[5] In the proof we shall obtain several relations among the various invariants of E, X, B and F. In conclusion, we shall formally extend (1.1) to a more general situation.

§ 2. Following J. H. C. Whitehead,[5] we have the exact sequence $\sum_*(K)$ and the partial exact sequence $\sum^*(K; G)$ for any simply connected CW-complex K and any Abelian group G:

$$\textstyle\sum_*(K) : \cdots \xrightarrow{j_*} H_{r+1}(K) \xrightarrow{d_*} \Gamma_r(K) \xrightarrow{i_*} \Pi_r(K) \xrightarrow{j_*} \cdots,$$

$$\textstyle\sum^*(K; G) : \cdots \xrightarrow{j^*} \Gamma^r(K; G) \xrightarrow{i^*} \Pi^r(K; G) \xrightarrow{d^*} H^{r+1}(K; G) \xrightarrow{j^*} \cdots$$

These are derived from the sequence

$$\cdots \xrightarrow{j_{r+1}} C_{r+1}(K) \xrightarrow{d_{r+1}} A_r(K) \xrightarrow{j_r} C_r(K) \xrightarrow{d_r} \cdots$$

and the G-dual

$$\cdots \xrightarrow{d_r^\#} C_r^\#(K;G) \xrightarrow{j_r^\#} A_r^\#(K;G) \xrightarrow{d_{r+1}^\#} C_{r+1}^\#(K;G) \xrightarrow{j_{r+1}^\#} \cdots,$$

where $C_{r+1}(K)=\pi_{r+1}(K^{r+1},K^r)$, $A_r(K)=\pi_r(K^r)$ if $r\geqq 2$, $C_2(K)$ is $\pi_2(K^2,K^1)$ made Abelian, d_{r+1}, j_r are the boundary and injection homomorphisms, and $C_r^\#(K;G)=\mathrm{Hom}(C_r(K),G)$, $A_r^\#(K;G)=\mathrm{Hom}(A_r(K),G)$, $d_{r+1}^\#$, $j_r^\#$ are the G-dual of $C_r(K)$, $A_r(K)$, d_{r+1}, j_r respectively.[6) Recall that

$$\Gamma_r(K)=j_r^{-1}(0),\quad \Pi_r(K)=A_r(K)/d_{r+1}C_{r+1}(K)$$
$$H_r(K)=Z_r(K)/\partial_{r+1}C_{r+1}(K);$$
$$\Gamma^r(K;G)=d_{r+1}^{\#-1}(0),\quad \Pi^r(K;G)=A_r^\#(K;G)/j_r^\# C_r^\#(K;G),$$
$$H^r(K;G)=Z^r(K;G)/\delta_r C_{r-1}^\#(K;G),$$

where $\partial_{r+1}=j_r\circ d_{r+1}$, $\delta_r=d_r^\#\circ j_{r-1}^\#$, $Z_r(K)=\partial_r^{-1}(0)$ and $Z^r(K;G)=\delta_{r+1}^{-1}(0)$. We notice that $\Pi_r(K)$ is isomorphic with $\pi_r(K)$ by the injection homomorphism. Using this isomorphism we make the identification

$$\Pi_r(K)=\pi_r(K).$$

Let $l_K^r: A_r(K)\longrightarrow \Pi_r(K)$ be the natural homomorphism, and let $\boldsymbol{l}_K^r\in \Pi^r(K;\pi_r(K))$ be the class containing $l_K^r\in A_r^\#(K,\pi_r(K))$. Then $l_K\circ d_{r+1}: C_{r+1}(K)\longrightarrow A_r(K)\longrightarrow \Pi_r(K)$ is trivial, and so $d_{r+1}^\# l_K^r=0$. Therefore we have

$$(2.1)\qquad l_K^r\in\Gamma^r(K,\pi_r(K)),\quad i^* l_K^r=\boldsymbol{l}_K^r.$$

Let $f: K\longrightarrow K'$ be a cellular map of K into a cell complex K'. Then f induces the homomorphisms $f'_\#:\ C_r(K)\longrightarrow C_r(K')$ and $f''_\#: A_r(K)\longrightarrow A_r(K')$, and further these induce the homomorphisms $f_*: \sum_*(K)\longrightarrow \sum_*(K')$ and $f^*:\ \sum^*(K';G)\longrightarrow \sum^*(K;G)$. As for the group $\Pi^r(K;G)$, we shall here note the following fact: Let $g: K^{r-1}\longrightarrow K'$ be a cellular map with a (cellular) extension $\tilde{g}: K^r\longrightarrow K'$. Then $\tilde{g}$ determines the homomorphism $\tilde{g}^*:\ \Pi^r(K';G)\longrightarrow \Pi^r(K;G)$. $\tilde{g}^*$ does not depend on the choice of an extension $\tilde{g}$, and further it is an invariant of the homotopy class of g.[5) Therefore we may write $g^*=\tilde{g}^*$.

Let (Y,Y') be a pair of topological spaces, and let $K(Y)$, $K(Y')$ be the singular polytopes of Y, Y'. Then $K(Y')$ is the closed subcomplex of the CW-complex $K(Y)$. Let $\kappa: K(Y)\longrightarrow Y$ be the projection. Then κ induces the isomorphism $\kappa_\#(\kappa^*)$ between the homotopy (singular cohomology) exact sequences for $(K(Y),K(Y'))$ and for (Y,Y'). By this isomorphism, we shall identify two exact sequences. Let $f:(Y,Y')\longrightarrow(Z,Z')$ be a continuous map. Then f induces a cellular map $K(f):(K(Y),K(Y'))\longrightarrow(K(Z),K(Z'))$, and the induced homomorphisms $K(f)_\#$ and $f_\#(f^*$ and $K(f)^*)$ are equivalent to each other by $\kappa_\#(\kappa^*)$.[7]

Let $M(Y)$ be the minimal subcomplex of the total singular complex of Y.[8] Then it is obvious that $M(Y)$ has a geometrical realization $\overline{M}(Y)$ in the singular polytope $K(Y)$. Further it can be seen[9] that $\overline{M}(Y)$ is a deformation retract of $K(Y)$, and that we can find the retraction $\varphi : K(Y) \longrightarrow \overline{M}(Y)$ which is cellular.

If Y is simply connected, $K(Y)$ and so $\overline{M}(Y)$ are also simply connected. Therefore we can consider the above sequences of J. H. C. Whitehead for $K(Y)$ and $\overline{M}(Y)$, and they are isomorphic by the homomorphism induced by φ.

§ 3. The fiber space (E, p, B) stated in § 1 is precisely as follows: The base space B is a space obtained by attaching cells of dimensionality $q+1, q+2, \ldots$ to X in such a way that $\pi_i(B)=0$ for $i \geqq q$, and the total space E is the space of paths $\{f : I \longrightarrow B, f(0) \in X, f(1) \in B\}$, where I is the unit interval. Further the projection $p : E \longrightarrow B$ is the map such that $p(f)=f(1)$ for all $f \in E$. Thus the fiber F is the space of paths $\{f : I \longrightarrow B; f(0) \in X, f(1)=x_0\}$. Notice that X is the subspace of B.

Let $\xi : F \subset E$ be the inclusion map, and let $\eta : X \longrightarrow E$ be a map such that $(\eta(x))(t)=x$ for $x \in X, t \in I$. Then ξ and η induce the isomorphisms

$$\xi_\# : \pi_q(F) \approx \pi_q(E), \quad \eta_\# : \pi_q(X) \approx \pi_q(E).$$

Since F is $(q-1)$-connected, we have the Hurewicz isomorphism: $\pi_q(F) \approx H_q(F)$. We shall use these isomorphisms to make the identifications

$$H_q(F)=\pi_q(F)=\pi_q(E)=\pi_q.$$

Then, the basic cohomology class $\boldsymbol{b}^q \in H^q(F, \pi_q)$ is the element which goes to the identical isomorphism by the natural homomorphism

$$H^q(F; \pi_q) \approx \mathrm{Hom}\,(H_q(F), \pi_q) = \mathrm{Hom}\,(H_q(F), H_q(F)).$$

Since the inclusion $\zeta : X \subset B$ induces the isomorphism $\zeta_\# : \pi_i(X) \approx \pi_i(B)$ for $i<q$, we may choose $M(X)$ and $M(B)$ as follows:

$$M(X) \subset M(B), \quad M(X)^{q-1}=M(B)^{q-1}.$$

Let $h' : \overline{M}(B)^{q-1} \longrightarrow \overline{M}(X)^{q-1} \subset K(X)$ be the identical map, and let $h=h' \circ (\varphi \mid K(B)^{q-1}) : K(B)^{q-1} \longrightarrow K(X)$. Then it follows[10] that h' has the cellular extension $\tilde{h}' : \overline{M}(B)^q \longrightarrow K(X)$ and that the secondary obstruction $\boldsymbol{c}^{q+1}(h') \in H^{q+1}(\overline{M}(B), \pi_q)$ is geometrically equivalent to the Eilenberg-MacLane invariant $\boldsymbol{k}_n^{q+1} \in H^{q+1}(\pi_n, n; \pi_q)$. Therefore, if we write $\boldsymbol{k}_n^{q+1} \in H^{q+1}(K(B); \pi_q)$ the element which corresponds to $\boldsymbol{k}_n^{q+1}$

by the natural isomorphism $H^{q+1}(K(B);\pi_q)\approx H^{q+1}(\pi_n, n; \pi)$, then we have

$$\boldsymbol{c}^{q+1}(h)=\bar{\boldsymbol{k}}_n^{q+1}.$$

Let $\boldsymbol{h}^q_{K(B)}\in \Pi^q(K(B);\pi_q)$ be a class containing the element $l^q_{K(X)}\circ \tilde{h}''_{\#}\in A^{\#}_q(K(B);\pi_q)$. Then it is obvious that

(3.1) $$h^*\boldsymbol{l}^q_{K(X)}=\boldsymbol{h}^q_{K(B)},$$

where $h^*:\Pi^q(K(X);\pi_q)\longrightarrow \Pi^q(K(B);\pi_q)$ is the homomorphism determined by h. Furthermore, since $c^{q+1}(\tilde{h})=\tilde{h}'_{\#}\circ d_{q+1}$ by the definition, we see that

(3.2) $$\bar{\boldsymbol{k}}_n^{q+1}=d^*_B\boldsymbol{h}^q_{K(B)},$$

where $d^*_B:\Pi^q(K(B);\pi_q)\longrightarrow H^{q+1}(K(B);\pi_q)$.

We have the commutative diagram

$$\begin{array}{ccccc}
\Pi^q(K(B);\pi_q) & \xrightarrow{\approx} & \Pi^q(\overline{M}(B);\pi_q) & \xleftarrow{\approx} & \Pi^q(K(B);\pi_q)\\
\downarrow K(\zeta)^* & & (h'^{-1})^*\downarrow\ \uparrow h'^* & & \uparrow h^*\\
\Pi^q(K(X);\pi_q) & \xrightarrow{\approx} & \Pi^q(\overline{M}(X);\pi_q) & \xleftarrow{\approx} & \Pi^q(K(X);\pi_q),
\end{array}$$

where all the horizontal arrows denote the isomorphisms induced by the inclusion maps, and h'^{-1} is the inverse map of h'. Therefore it holds that $h^{*-1}=K(\zeta)^*$ and so from (3.1)

(3.3) $$\boldsymbol{l}^q_{K(X)}=K(\zeta)^*\boldsymbol{h}^q_{K(B)}.$$

Since $p\circ\eta=\zeta$, we have the commutative diagram

$$\begin{array}{ccc}
\Pi^q(K(E);\pi_q) & \xleftarrow{K(p)^*} & \Pi^q(K(B);\pi_q)\\
\downarrow K(\eta)^* & \swarrow K(\zeta)^* & \\
\Pi^q(K(X);\pi_q) & & .
\end{array}$$

Further it is obvious that $K(\eta)^*$ is isomorphic and $\boldsymbol{l}^q_{K(X)}=K(\eta)^*\boldsymbol{l}^q_{K(E)}$. Therefore it follows from (3.3) that

(3.4) $$\boldsymbol{l}^q_{K(E)}=K(p)^*\boldsymbol{h}^q_{K(B)}.$$

Since $H_q(F)/\sum_q(F)=0$, we have $j_qA_q(K(F))=Z_q(K(F))$.[3)5)] Since further $Z_q(K(F))$ is free Abelian, there exists a homomorphism $\lambda: Z_q(K(F))\longrightarrow A_q(K(F))$ such that

(3.5) $$j_F\circ\lambda: Z_q(K(F))\longrightarrow C_q(K(F))$$

is the inclusion.

Let $u\in A^{\#}_q(K(B);\pi_q)$ be a representative of $\boldsymbol{h}^q_{K(B)}$. Then it follows from (3.4) that $u\circ K(p)_{\#}\in A^{\#}_q(K(E);\pi_q)$ is a representative of $\boldsymbol{l}^q_{K(E)}$. On the other hand, we see from (2.1) that $l^q_{K(E)}$ is a representative of $\boldsymbol{l}^q_{K(E)}$. Therefore it follows from the definition of $\Pi^q(K(E);\pi_q)$ that there exists an element $v\in C^{\#}_q(K(E);\pi_q)$ such that

(3.6) $$v\circ j_E=l^q_{K(E)}-u\circ k(p)_{\#},$$

where $j_E=j_q: A_q(K(E))\longrightarrow C_q(K(E))$ is the injection.

Consider the commutative diagram

$$\begin{array}{ccccc} C_{q+1}(K(F)) & \xrightarrow{\partial_{q+1}} & C_q(K(F)) & \xrightarrow{K(\xi)'_{\#}} & C_q(K(E)) \\ {\scriptstyle d_{q+1}}\downarrow & Z_q(K(F)) & \uparrow{\scriptstyle j_F} & & \uparrow{\scriptstyle j_E} \\ & \xrightarrow{\lambda} & A_q(K(F)) & \xrightarrow{K(\xi)''_{\#}} & A_q(K(E)), \end{array}$$

and notice that

$$l^q_{K(E)}\circ K(\xi)''_{\#}=l^q_{K(F)},\quad K(p)''_{\#}\circ K(\xi)''_{\#}=0.$$

Then we see from (3.6) and (2.1)

$$\begin{aligned} v\circ K(\xi)'_{\#}\circ\partial_{q+1}&=v\circ j_E\circ K(\xi)''_{\#}\circ d_{q+1}\\ &=(l^q_{K(E)}-u\circ K(p)_{\#})\circ K(\xi)''_{\#}\circ d_{q+1}\\ &=l^q_{K(F)}\circ d_{q+1}-u\circ K(p)_{\#}\circ K(\xi)''_{\#}\circ d_{q+1}=0, \end{aligned}$$

and so $v\circ K(\xi)'_{\#}\in Z^q(K(F);\pi_q)$. Moreover we have for any element $z\in Z_q(K(F))$

$$\begin{aligned} v\circ K(\xi)'_{\#}(z)&=v\circ K(\xi)'_{\#}\circ j_F\circ\lambda(z)\\ &=v\circ j_E\circ K(\xi)''_{\#}\circ\lambda(z) \qquad (3.7)\\ &=(l^q_{K(E)}-u\circ K(p)_{\#})\circ K(\xi)''_{\#}\circ\lambda(z)=l^q_{K(F)}\circ\lambda(z). \end{aligned}$$

Since it holds obviously

$$\nu\circ j_F=l^q_{K(F)},$$

where $\nu: Z_q(K(F))\longrightarrow H_q(K(F))=\pi_q$ is the natural homomorphism, we have from (3.5) and (3.7)

$$v\circ K(\xi)'_{\#}(z)=l^q_{K(F)}\circ\lambda(z)=v\circ j_F\circ\lambda(z)=\nu(z).$$

Therefore we see from the definition of $\boldsymbol{b}^q$ that the cocycle $v\circ K(\xi)'_{\#}$ is a representative of $\boldsymbol{b}^q$. Thus $\delta^*\boldsymbol{b}^q$ is the class of $H^{q+1}(K(E),K(F);\pi_q)$ containing $\delta_E v$. However

$$\begin{aligned} \delta_E v&=v\circ\partial_E=v\circ j_E\circ d_E\\ &=(l^q_{K(E)}-u\circ K(p)_{\#})\circ d_E\\ &=-u\circ K(p)''_{\#}\circ d_E \qquad \text{(see (2.1))}\\ &=-u\circ d_B\circ K(p)'_{\#}. \end{aligned}$$

Therefore

$$\delta^*\boldsymbol{b}^q=-p^*\circ d_B^*\circ\boldsymbol{h}^q_{K(B)}$$

and it follows from (3.2) that

$$\delta^*\boldsymbol{b}^q=-p^*\bar{\boldsymbol{k}}_n^{q+1}.$$

Namely we have (1.1).

§ 4. Let X be a 1-connected space, and write briefly $\pi_j(X)=\pi_j$ $(j=2,3,\dots)$. Consider a space B obtained by attaching cells of dimensionality $r+1$, $r+2,\dots$ to X in such a way that $\pi_i(B)=0$ for $i\geqq r$, and construct a fiber space (E,p,B) by the same way as in § 3. Then we can see easily that the fiber F is an $(r-1)$-connected space, and the homology group of B is naturally equivalent to that of the Postnikov model complex $K_{r-1}=K(1, \pi_2, \ldots, \pi_{r-1}; 0, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_{r-1})$, where $\boldsymbol{k}_i$ denotes the Postnikov invariant of the space X.[11] Let $\bar{\boldsymbol{k}}_{r-1} \in H^{r+1}(B; \pi_r)$ be the geometrical equivalent of the element $\boldsymbol{k}_{r-1} \in H^{r+1}(K_{r-1}; \pi_r)$. Then we have by the similar arguments as in § 3

$$\tau \boldsymbol{b}^r = -\bar{\boldsymbol{k}}_{r-1},$$

where $\boldsymbol{b}^r$ is the basic cohomology class of F, and $\tau : H^r(F; \pi_r) \longrightarrow H^{r+1}(B; \pi_r)$ is the transgression.

References

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2) J-P. Serre: Ann. Math., **54**, 425-505 (1951).

3) J. H. C. Whitehead: Ibid., **54**, 68-84 (1951).

4) S. Eilenberg and S. MacLane: Ibid., **51**, 514-533 (1950).

5) J. H. C. Whitehead: Proc. Lond. Math. Soc., **12**, 385-416 (1953). He does not necessarily assume that K is simply connected.

6) See § 5 in 5) for the definition when $r<2$. j_2 is the composite of the injection $A_2(K) \to \pi_2(K^2, K^1)$ followed by the natural homomorphism $\pi_2(K^2, K^1) \to C_2(K)$.

7) For the detailed accounts, see the followings: J. B. Giever: Ann. Math., **51**, 178-191 (1950); S. T. Hu: Osaka Math. J., **2**, 165-209 (1950); J. H. C. Whitehead: Ann. Math., **52**, 51-110 (1950).

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9) See p. 504 of 8).

10) See p. 519 of 4).

11) M. M. Postnikov: Doklady Akad. Nauk SSSR., **76**, 359-362 (1951); ibid., **76**, 789-791 (1951). See also the report of P. J. Hilton (mineographed) and the paper of K. Mizuno (to appear in J. Inst. Polyt., Osaka City Univ., **5** (1954)).