69. On Ergodic Theorems

By Shigeru TSURUMI

Mathematical Institute, Tokyo Metropolitan University

(Comm. by Z. SUETUNA, M.J.A., May 13, 1954)

1. Introduction. Let $(X, \mathfrak{B}, m)$ denote a measure space such that X is a set, $\mathfrak{B}$ is a Borel field of subsets of X, and m is a σ-finite measure defined on $\mathfrak{B}$. Let T be a single-valued, measurable and non-singular transformation of X into itself. The measurability and non-singularity of the transformation are used in the sense of Ryll-Nardzewski [3].[1] In the following, all sets in consideration are supposed $\mathfrak{B}$-measurable.

We define the following statements.

(I)   *There exists a constant K such that for any set A of positive measure*

$$0 < \limsup_{n} \frac{1}{n} \sum_{i=0}^{n-1} m(T^{-i}A) \leq K \cdot m(A).$$

(I')   *There exists a constant K such that for any set A*

$$\limsup_{n} \frac{1}{n} \sum_{i=0}^{n-1} m(T^{-i}A) \leq K \cdot m(A).$$

(II)   *There exist a sequence of sets $\{X_j\}$ and a constant K such that*

$$X_1 \subset X_2 \subset \cdots, \quad X = \bigcup_{j=1}^{\infty} X_j, \quad m(X_j) < \infty \quad (j = 1, 2, \ldots),$$

and for any set A of positive measure

$$0 < \sup_{j} \limsup_{n} \frac{1}{n} \sum_{i=0}^{n-1} m(X_j \cap T^{-i}A) \leq K \cdot m(A) \quad (j = 1, 2, \ldots).$$

(II')   *There exist a sequence of sets $\{X_j\}$ and a constant K such that*

$$X_1 \subset X_2 \subset \cdots, \quad X = \bigcup_{j=1}^{\infty} X_j, \quad m(X_j) < \infty \quad (j = 1, 2, \ldots),$$

and for any set A

$$\limsup_{n} \frac{1}{n} \sum_{i=0}^{n-1} m(X_j \cap T^{-i}A) \leq K \cdot m(A) \quad (j = 1, 2, \ldots).$$

(B)   *For any function $f \in L(X, \mathfrak{B}, m)$ the limit*

$$\tilde{f}(x) = \lim_{n} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$$

exists almost everywhere (m) and $\tilde{f} \in L(X, \mathfrak{B}, m)$.

---

1) Numbers in square brackets refer to the references at the end of this paper.

In case m is finite, N. Dunford and D. S. Miller [1] have given a necessary and sufficient condition[2] that Neumann's ergodic theorem holds and led the statement (B) from this condition. Hereafter F. Riesz [4] has given another proof of the latter and proved that, in case m is not finite, the above condition with a certain additional condition implies (B). Recently C. Ryll-Nardzewski [3] has shown that the statement (II′) is equivalent to (B) and, in case m is finite, (I′), (II′) and (B) are equivalent to each other. However a part of the former is not quite right in case m is not finite. In fact we can construct a σ-finite (but not finite) measure space and a transformation for which (II′) holds and (B) does not hold.

The main purpose of this paper is to show that each of (I) and (II) implies (B), and that (II′) does not necessarily imply (B). The detail of this paper will appear in the Tôhoku Mathematical Journal, so that we shall state the outline.

2. Generalization of Birkhoff's Ergodic Theorem. Let $\varDelta(A)$ $=\varDelta(A, \{A_k\}_{k=1,2,\ldots})$ denote a decomposition of a set A such that

$$A = \bigcup_{k=1}^{\infty} A_k, \quad A_k \cap A_l = 0 \quad (k \neq l).$$

Let us put for every set A

$$(1) \qquad a(A) = \sup_{(A)} \sum_{(A:k)} \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} m(T^{-i}A_k),$$

where $\sup_{(A)}$ denotes the supremum for all decompositions $\varDelta(A)$ of A and $\sum_{(A:k)}$ means to sum up with respect to the sets A_k's of the decomposition $\varDelta(A) = \varDelta(A, \{A_k\}_{k=1,2,\ldots})$.

Lemma 1. *If the statement* (I′) *holds, the non-negative set function* a *defined by* (1) *has the following properties:*

(i) a *is finitely additive;*

(ii) $a(A) \leqq a(T^{-1}A)$ *for any set* A *of finite measure;*

(iii) $a(A) \leqq K \cdot m(A)$ *for any set* A*;*

(iv) $a(A) \geqq \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} m(T^{-i}A)$ *for any set* A.

C. Ryll-Nardzewski [3] has proved the following

Lemma 2. *Let* $(X', \mathfrak{B}', \mu)$ *be a measure such that* μ *is a measure defined on* $\mathfrak{B}'$, *and* $\tilde{T}$ *be a transformation of* $L(X', \mathfrak{B}', \mu)$ *into itself which has the following properties:*

---

2) The condition reads as follows: there exists a constant K such that for any set A

$$\frac{1}{n} \sum_{i=0}^{n-1} m(T^{-i}A) \leqq K \cdot m(A) \qquad (n = 1, 2, \ldots).$$

(i) if $f(x)=g(x)$ almost everywhere (μ), $\widetilde{T}f(x)=\widetilde{T}g(x)$ almost everywhere (μ);

(ii) $\widetilde{T}$ is additive and homogeneous;

(iii) if $f(x)$ is positive almost everywhere (μ), $\widetilde{T}f(x)$ is also. Then $\widetilde{T}$ is a linear operator of $L(X', \mathfrak{B}', \mu)$ into itself.

By use of Lemmas 1 and 2 we can prove the following theorem which is a generalization of Birkhoff's ergodic theorem.

Theorem 1. The statement (I) implies the statement (B). If m is finite, the statements (I), (I') and (B) are equivalent to each other.

The following result obtained by F. Riesz [4] follows immediately from Theorem 1.

Corollary. If there exist two constants K_1 and K_2 such that for any set A

$$K_1 \cdot m(A) \leq \frac{1}{n} \sum_{i=0}^{n-1} m(T^{-i}A) \leq K_2 \cdot m(A) \quad (n=1, 2, \ldots),$$

then the statement (B) holds.

The following result obtained by F. Riesz [4] follows immediately from Theorem 1.

Theorem 2. The statement (II) implies the statement (B).

Further following two theorems hold.

Theorem 3. Let us assume that there exists a finite invariant measure equivalent to m. Then the statements (II), (II') and (B) are equivalent to each other.

Theorem 4. Let us assume that X is the union of countable invariant subsets of finite measure. Then the statements (II), (II') and (B) are equivalent to each other.

3. Counter Examples. By constructing counter examples we can show the following facts:

1° the statement (B) does not necessarily imply the statement (I');

2° the statement (B) does not necessarily imply the statement (II);

3° if the assumption in Theorem 3 is omitted, the statement (II') does not necessarily imply the statement (B).

4° if the assumption of Theorem 4 is omitted, the statement (II') does not necessarily imply the statement (B).

We are contented here with showing that the statement (II') does not necessarily imply the statement (B). Our example is a modification of that constructed by P. R. Halmos [2; pp. 743–744].

He has constructed a measure space $(X, \mathfrak{B}, \mu)$ and a transformation T as follows. A collection of linear intervals $J_{n,k}$'s in the (s, t)-plane is defined by

$$J_{n,k}=\left\{(s, t) ; \ \frac{1}{2^{n+1}} \leq s < \frac{1}{2^n}\right\} \quad \left(\begin{matrix} k=0, 1, \ldots, \ 2^{n+1}-1 ; \\ n=0, 1, 2, \ldots \end{matrix}\right).$$

Let $(X, \mathfrak{B}, \mu)$ denote a measure space such that X is the union of all $J_{n,k}$'s, μ is the class of all Lebesgue measurable subsets of X, and m is the ordinary linear Lebesgue measure. Then it is easy to see that there exists a one to one, measurable, measure-preserving (with respect to μ) and ergodic transformation T_0 of $\bigcup\limits_{n=0}^{\infty} J_{n,0}$ onto itself. We define now a transformation T of X onto itself by

$$T(s, t) = (s, t+1), \quad (s, t) \in J_{n,k} \quad \begin{pmatrix} k=0, 1, \ldots, 2^{n+1}-2; \\ n=0, 1, 2, \ldots \end{pmatrix},$$

$$= T_0(s, 0), \quad (s, t) \in J_{n, 2^{n+1}-1} \quad (n=0, 1, 2, \ldots).$$

Next, we introduce a new measure m on $\mathfrak{B}$ by

$$m(A) = \mu(A)/[2(n+1)^2-1], \quad A \subset J_{n, 2^{n+1}-1} \quad (n=0, 1, 2, \ldots),$$

$$= \mu(A), \quad A \subset J_{n,k} \begin{pmatrix} k=0, 1, \ldots, 2^{n+1}-2; \\ n=0, 1, 2, \ldots \end{pmatrix}.$$

Then we can show that

(i) m is a σ-finite (not finite) measure on $\mathfrak{B}$ equivalent to μ;

(ii) T is a one to one, measurable, non-singular and ergodic transformation of X onto itself;

(iii) the statement (II') holds.

Finally we get

(iv) the statement (B) does not hold.

In fact, if we define a function f by

$$f((s, t)) = 2^{n+1}[2(n+1)^2-1]/2(n+1)^2, \quad (s, t) \in J_{n, 2^{n+1}-1} \quad (n=0, 1, 2, \ldots),$$

$$= 2^{n+1}/2(n+1)^2(2^{n+1}-1), \quad (s, t) \in J_{n,k} \begin{pmatrix} k=0, 1, \ldots, n^{n+1}-2; \\ n=0, 1, 2, \ldots \end{pmatrix},$$

then we can prove that

$$\lim_{n} \sup \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(s, t)) = 1$$

almost everywhere (m). On the other hand, if we suppose that (B) holds, then the limit

$$\lim_{n} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(s, t))$$

exists almost everywhere (m) and the limit function vanishes almost everywhere (m) on account of the properties of μ, m and T. This is a contradiction, so that the statement (B) does not hold.

References

[1] N. Dunford and D. S. Miller: On the ergodic theorems, Trans. Amer. Math. Soc., **60**, 538-549 (1946).

[2] P. R. Halmos: Invariant measures, Ann. Math., **48**, 735-754 (1947).

[3] C. Ryll-Nardzewski: On the ergodic theorems (I) (Generalized ergodic theorems), Studia Math., **12**, 65-73 (1951).

[4] F. Riesz: On a recent generalisation of G. D. Birkhoff's ergodic theorem, Acta Szeged., **11**, 193-200 (1946-48).