92. A Proof for a Theorem of M. Nakaoka

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1. Let X be a simply connected topological space with vanishing homotopy groups $\pi_i(X)$ for $i<n$, $n<i<q$ and $q<i$. Then M. Nakaoka[1] proved that the transgression τ in the Cartan-Serre fiber space associated with X and the geometrical realization $\bar{\boldsymbol{k}}_n^{q+1}$ of the Eilenberg-MacLane invariant $\boldsymbol{k}_n^{q+1}$ are related as follows:

(1) $$\tau\boldsymbol{b}=-\bar{\boldsymbol{k}}_n^{q+1},$$

where $\boldsymbol{b}$ is the basic cohomology class of the fiber.

The purpose of this note is to construct a singular structure of an arbitrary fiber space (E, p, B) satisfying

(2) (i) the total space E is a simply connected space with vanishing homotopy groups $\pi_i(E)$ for $i>q$ with a base point e_0,

(ii) the base space B is a space with vanishing homotopy groups $\pi_i(B)$ for $i\geqq q$ with a base point $b_0=p(e_0)$,

(iii) the projection $p:E\longrightarrow B$ induces the isomorphisms $\pi_i(E)\approx\pi_i(B)$ for $i<q$,

(iv) the fiber $F=p^{-1}(b_0)$ is a space with a base point e_0.

And, as an application, we shall give a proof of the similar relation as (1) in an arbitrary fiber space satisfying (2) about the Postnikov invariant.[2]

This paper makes full use of the results and terminologies of the preceding paper by the author.[3]

2. Let Y be a topological space. A singular n-simplex T of Y is a function $T(x_0,\dots,x_n)\in Y$ defined for $0\leq x_i, x_0+x_1+\cdots+x_n=1$. For any element $\beta=\sum_j m_j\beta_j$ of $K_r(n)$, the β-face T_β of T is an r-chain defined as

$$T_\beta=\sum_j m_j T_{\beta_j},\quad T_{\beta_j}(x_0,\dots,x_r)=T(y_0,\dots,y_n),$$

where $y_i=0$ if $i\neq\beta_j(k)$ for all $k=0,\dots,r$, and $y_i=\sum_k x_k$ for $\beta_j(k)=i$. In particular, the ε^i-face of T will be denoted simply by $T^{(i)}$ and is called the i-th face.

1) M. Nakaoka: Transgression and the invariant $\boldsymbol{k}_n^{q+1}$, Proc. Japan Acad., **30**, 363-368 (1954).

2) Refer 3). Originally reported in the Math. Reviews, **13** (1952). (M. M. Postnikov: Doklady Akad. Nauk URSS., **76**, 359-362 (1951); ibid., **76**, 789-791 (1951)).

3) K. Mizuno: On the minimal complexes, Jour. Inst. Polytech., Osaka City Univ., **5**, 41-51 (1954).

For our future convenience we shall fix a homeomorphism h_n ($n=1, 2, \dots$) of n-simplex into the face of the $(n+1)$-prism excepting the lower base as follows:

$$h_n(x_0, \dots, x_n)=(y_0, \dots, y_n, t),$$

where $t=\min\{1, (n+2)\min_{0\leq i\leq n} x_i\}$ and $y_i=\{(n+2)x_i-t\}/\{(n+2)-(n+1)t\}$. And, we write a singular n-cylinder f of Y for a function $f(x_0, \dots, x_n, t) \in Y$ defined for $x_0+x_1+\cdots+x_n=1$, $\min_{0\leq i\leq n} x_i=0$ and $0\leq t\leq 1$, with its partial map $f_1=f|_{t=1}$.

3. For any singular n-simplex T' of B, we can select a singular n-simplex T'_* of B which is compatible and homotopic with T' such that

$$T'_*(x_0, \dots, x_n)=b_0 \quad \text{if } (n+2)\min_{0\leq i\leq n} x_i \geq 1,$$

then, in the following, we write $M(B)$ for the minimal subcomplex whose simplexes satisfying this condition.

Let us define an FD-map $p_*^{-1}: M(B) \longrightarrow S(E)$ in dimension $\leq q$ as follows:

Let T' is a singular n-simplex of $M(B)$, we shall define a singular cylinder $f(T')$ as the partial map of $T'h_n^{-1}$, and in particular $f(T')(1)=b_0$ if $n=0$.

Then, by the covering homotopy theorem, we have a singular cylinder $\bar{f}(T')$ of E such as $p\bar{f}(T')=f(T')$ in dimension $\leq q$ inductively. Especially, we choose $f(T')$ to be the collapsed one if T' is collapsed.

Now, the partial map $\bar{f}(T')_1$ induces an element of $\pi_{n-1}(F)$, and if $n\leq q$, by our original assumption, we can extend the map $\bar{f}(T')_1$ over the upper base of the prism. Especially, we choose a collapsed singular n-simplex as this extension if T' is collapsed.

If we combine the singular cylinder $\bar{f}(T')$ with this extension, we have a map $\bar{f}_*(T')$ of the face of the $(n+1)$-prism excepting the lower base into the space E, consequently we have a singular n-simplex of $S(E)$, denoted by $p_*^{-1}(T')$, as $p_*^{-1}(T')=\bar{f}_*(T')h_n$. Thus $pp_*^{-1}(T')=T'$.

This map p_*^{-1} induces a map of the minimal complex $M(B)$ isomorphically onto a minimal complex $M(E)$ of E in dimension $<q$ since the projection p satisfies the condition (2) (iii). And, for each q-simplex T'_q of $M(B)$ there is at least one q-simplex T_q of $M(E)$ such that $pT_q=T'_q$. Any two such simplexes T_q are compatible. One of these simplexes T_q will be selected and denoted by $p_*^{-1}(T'_q)$. Thus $pp_*^{-1}(T'_q)=T'_q$. For the collapsed q-simplex T'_q, we choose $p_*^{-1}(T'_q)$ to be the collapsed q-simplex in $M(E)$.

On the other hand, for any singular n-simplex T_n of $M(E)$, we shall define a singular n-simplex $T'_n = pT_n$ and an element $\psi_n = \psi(T_n)$ of $F_n(\pi_q, q)$[4] such as

$$\psi_n(\beta) = d(p_*^{-1}(pT_n)_\beta, T_{n\cdot\beta}) \quad \text{for any element } \beta \text{ of } K_q(n),$$

and, we have an FD-map

$$p_* : M(E) \longrightarrow M(B) \times F(\pi_q, q).$$

If we attempt to continue the definition of p_*^{-1} for $(q+1)$-simplexes T'_{q+1} of $M(B)$, we can only go as far as to define a map $\bar{f}(T_{q+1})_1$ and we have a cochain $\bar{k}_{q-1}$ defined by

$$\bar{k}_{q-1}(T'_{q+1}) = c(\bar{f}(T'_{q+1})_1).$$

Now, each element (T'_n, ψ_n) in the image of p_* satisfies the condition

$$(3) \qquad \sum_{i=0}^{q+1}(-1)^i \psi_n(\gamma \varepsilon_{q+1}^i) + \bar{k}_{q-1}(T'_{n\cdot\gamma}) = 0$$

for any element γ of $K_{q+1}(n)$. Conversely, for any element (T'_n, ψ_n) of the cartesian product $M(B) \times F(\pi_q, q)$ satisfying the condition (3), there exists a unique singular simplex of $M(E)$, denoted by $p_*^{-1}(T'_n, \psi_n)$.[5] Thus $p_* p_*^{-1}(T'_n, \psi_n) = (T'_n, \psi_n)$.

It is obvious that $\bar{k}_{q-1}$ is a cocycle of $Z^{q+1}(B; \pi_q)$ and its cohomology class $\bar{\boldsymbol{k}}_{q-1}$ is uniquely determined only by the fiber space (E, p, B). And, if E is of the same homotopy type with X in **1**, $\bar{\boldsymbol{k}}_{q-1}$ is a geometrical realization of the Eilenberg-MacLane invariant $\boldsymbol{k}_n^{q+1}$ associated with our fiber space.

4. Consider in our fiber space the transgression

$$\tau = p^{*-1}\delta^* : H^q(F; \pi_q) \longrightarrow H^{q+1}(B; \pi_q),$$

where $\delta^* : H^q(F; \pi_q) \longrightarrow H^{q+1}(E, F; \pi_q)$ is the coboundary homomorphism, and $p^* : H^{q+1}(B; \pi_q) \longrightarrow H^{q+1}(E, F; \pi_q)$ is the isomorphism induced by p.

Since any $(q-1)$-dimensional face of any singular q-simplex of $M(E) \cap S(F)$ is collapsed, the basic cohomology class $\boldsymbol{b} \in H^q(F; \pi_q)$ is represented by a cocycle b which is defined as

$$b(T_q) = d(p_*^{-1}pT_q, T_q) = \psi(T_q)(\varepsilon_q)$$

for any T_q of $M(E) \cap S(F)$.

Let us define a cochain u of $C^q(E, F; \pi_q)$ as follows:

$$\begin{aligned} u(T_q) &= \psi(T_q)(\varepsilon_q) && \text{if } T_q \in M(E) - S(F) \\ &= 0 && \text{if } T_q \in M(E) \cap S(F). \end{aligned}$$

4) For the sake of brevity, we write $\pi_q = \pi_q(E) = \pi_q(F)$.

5) For example, for any element (T'_q, ψ_q) of $M(B) \times F(\pi_q, q)$, there exists a unique singular simplex T_q of $M(E)$, compatible with $p_*^{-1}(T'_q)$ and satisfies $d(p_*^{-1}(T'_q), T_q) = \psi_q(\varepsilon_q)$.

The coboundary homomorphism

$$\delta^{\#}: Z^q(F; \pi_q) \longrightarrow Z^{q+1}(E, F; \pi_q)$$

is calculated as follows:

$$\delta^{\#} v(T_{q+1}) = \sum_{i \in I} (-1)^i v(T_{q+1}^{(i)})$$

for any cocycle $v \in Z^q(F; \pi_q)$ and for any singular $(q+1)$-simplex $T_{q+1} \in M(E)$ where $I = \{i; 0 \leq i \leq q+1 \text{ and } T_{q+1}^{(i)} \in S(F)\}$.

Then, it follows from (3) that

$$(4) \qquad p^{\#}\bar{k}_{q-1}(T_{q+1}) + \delta^{\#} b(T_{q+1}) + \delta_r u(T_{q+1})$$
$$= \bar{k}_{q-1}(pT_{q+1}) + \sum_{i=0}^{q+1} (-1)^i \psi(T_{q+1})(\varepsilon_{q+1}^i) = 0$$

where $p^{\#}: Z^{q+1}(B; \pi_q) \longrightarrow Z^{q+1}(E, F; \pi_q)$ is the homomorphism induced by p, and $\delta_r: C^q(E, F; \pi_q) \longrightarrow C^{q+1}(E, F; \pi_q)$ is the relative coboundary homomorphism.

The similar relation as (1) about the Postnikov invariant can be proved as the immediate consequence of (4).