120. A Note on the Set of Logarithmic Capacity Zero

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1. Let F be an open Riemann surface and let G be a domain on F whose relative boundary C consists of at most an enumerable number of analytic curves being compact or non-compact and clustering nowhere in F. Denote by Γ the ideal boundary of G which is a part of the ideal boundary of F. From G we delete a simply connected domain Δ which is compact with respect to the domain G and whose boundary consists of an analytic curve γ and we denote by G_0 the resulting domain.

We can construct an open Riemann surface $\hat{G}$ by the process of symmetrization. There is given an indirectly conformal mapping of $\hat{G}$ on itself which leaves every point on C fixed.

2. The result which we shall prove here is as follows:

Theorem 1. *Suppose that there exists a non-constant, single-valued and harmonic function $u(p)$ in $G (p \in G)$ satisfying the following conditions:*

(i) *$u(p)$ is continuous on $G \smile C$ and $\lim_{p \to \Gamma} u(p) = +\infty$, where the point p varies on $G \smile C$;*

(ii) *for the conjugate function $v(p)$ of $u(p)$, there exists a constant M such that*

$$\theta(r) = \int_{\Gamma_r} dv \leqq M,$$

where Γ_r is any niveau curve $u(p) = r$.

Then $\hat{G}$ has a null boundary.

Proof. Let $\tilde{\Delta}$ be the image of Δ on $\hat{G}$ under the indirectly conformal mapping of $\hat{G}$ on itself and let $\tilde{\gamma}$ be the boundary of $\tilde{\Delta}$. Deleting Δ and $\tilde{\Delta}$ from $\hat{G}$, we get a domain $\hat{G}_0$ which is bounded by γ, $\tilde{\gamma}$ and the ideal boundary of $\hat{G}$. We choose an exhaustion $\{\hat{G}_n\}$ $(n = 1, 2, \ldots)$ of $\hat{G}_0$ such that, for each n, the boundary of $\hat{G}_n$ consists of γ, $\tilde{\gamma}$ and γ_n, where γ_n is symmetric with respect to C. Denote by $\omega_n(p)$ $(p \in \hat{G}_n)$ the harmonic measure of γ_n with respect to $\hat{G}_n$. It is obvious from the symmetric property of γ_n that $\omega_n(p)$ is symmetric; more precisely, if we denote by $\tilde{p}$ the image of a point

$p \in G$ under the indirectly conformal mapping of $\hat{G}$ on itself $\omega_n(p) = \omega_n(\tilde{p})$. Since it is obvious that $\omega_n(p) \geqq \omega_{n+1}(p)$ for each n, the sequence $\omega_n(p)$ has a uniquely determined limiting function $\omega(p)$. From the fact that $\omega_n(p) = \omega_n(\tilde{p})$, we can see that for any $p \in G_0$

$$(1) \qquad \omega(p) = \omega(\tilde{p}).$$

This function $\omega(p)$ is nothing but the harmonic measure of the ideal boundary of $\hat{G}$ with respect to the domain $\hat{G}_0$ and is harmonic on γ, $\tilde{\gamma}$ and C. Further it is easily seen from (1) that the normal derivative $\frac{\partial \omega}{\partial \nu}$ vanishes at every point on C. It is well known that $\hat{G}$ has a null boundary if and only if $\omega(p)$ is identically equal to zero. Hence, to establish our theorem, it is sufficient to prove that $\omega(p) \equiv 0$ under the condition of our theorem.

Consider the function $u(p)$ satisfying the conditions of the theorem. Let us denote by ρ_0 the maximum of the function $u(p)$ on γ. For any $r \geqq r_0 = \rho_0 + 1$, the niveau curve $\Gamma_r : u(p) = r$ separates the ideal boundary Γ of G from γ. We shall denote by G_r the open subset of G_0 which is the set of points satisfying $u(p) < r$. Since $u(p)$ is not equal to $r_0 + 1$ at every point of γ, G_r consists of at most an enumerable number of domains. Denoting by $D(r)$ the Dirichlet integral of $\omega(p)$ taken over G_r, we have

$$D(r) = \int_{C_r \cup \Gamma_r \cup \gamma} \omega \frac{\partial \omega}{\partial \nu} ds,$$

where C_r is a part of boundary of G_r contained in C, ν is the inner normal and the integral is taken in the positive sense with respect to G_r. Since $\omega(p) = 0$ on γ and $\frac{\partial \omega}{\partial \nu} = 0$ on C as stated above, we get

$$D(r) = \int_{\Gamma_r} \omega \frac{\partial \omega}{\partial u} dv.$$

By the Schwarz inequality, we can easily obtain

$$(D(r))^2 \leqq \int_{\Gamma_r} \omega^2 dv \int_{\Gamma_r} \left(\frac{\partial \omega}{\partial u} \right)^2 dv$$
$$\leqq M \frac{dD(r)}{dr},$$

whence follows that

$$dr \leqq M \frac{dD(r)}{(D(r))^2}.$$

Integrating both sides, we have

$$r - r_0 = M \int_{r_0}^{r} \frac{dD(r)}{(D(r))^2} = M \left[\frac{1}{D(r_0)} - \frac{1}{D(r)} \right]$$
$$\leqq M \frac{1}{D(r_0)}.$$

Since the left hand side diverges, $D(r_0)$ must be zero, which shows that $\omega(p)$ identically equals to the constant zero. Thus our theorem is proved.

3. By the same argument as in the above proof, we have the following

Theorem 2. *Under the same condition as in Theorem 1, there exists no non-constant, single-valued, bounded and harmonic function in G whose normal derivative vanishes at every point on C.*

This theorem is also proved by using Theorem 1 and a result which was proved already by the one of the authors [2].

4. Let E be the closed set lying on the unit circle $|z|=1$ and not being identical to the whole circle $|z|=1$ and let C be the complementary set of E with respect to the circle $|z|=1$. Consider the complementary domain of E with respect to the whole z-plane as the open Riemann surface. Then the unit circular disc $G:|z|<1$ is a non-compact domain with the relative boundary C.

It is well known that, if E is of measure zero, there exists a non-constant harmonic function $u(z)$ in the disc G satisfying the condition (i) of Theorem 1. (Cf. Noshiro [4].)

We notice that the surface $\hat{G}$ obtained by the symmetrization of the disc G with respect to C is of genus zero. Since $\hat{G}$ coincides with the complementary domain of E with respect to the whole z-plane, we have the following corollary which was conjectured by Professor Noshiro.

Theorem 3. *Suppose that there exists such a function $u(z)$ as stated above whose conjugate harmonic function $v(z)$ satisfies the condition* (ii) *of Theorem 1. Then the set E is of logarithmic capacity zero. The converse is also true.*

The proof of the first part was just stated above. Since G is simply connected in this case, this can be also established by Theorem 2 and a result proved already in [2]. The second part is proved by the fact that, if E is of logarithmic capacity zero, there exists the Evans' potential on the complementary domain of E. (Cf. [4].)

5. **Remark.** Professor Noshiro pointed out the fact that the same results as Theorems 1, 2 and 3 are also obtained whenever the condition (ii) is replaced by the condition that the integral

$$\int^{r} \frac{dr}{\theta(r)}$$

diverges as $r \to +\infty$. This can be easily seen by the argument used in the proof of Theorem 1.

Theorem 3 is closely related to results of Mori [3] and Kaplan [1].

References

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