116. Note on Generalized Uniserial Algebras. II

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(Comm. by K. SHODA, M.J.A., July 12, 1954)

Let A be an associative (and finite dimensional) algebra with a unit and K be a ground field. In the previous paper[1] one of the authors proved that the *absolutely generalized uniserial* algebras, i.e. the generalized uniserial algebras which remain so after any coefficient field extension, are the direct sum of two subalgebras, one of which is a generalized uniserial algebra with the separable residue class algebra over its radical and the other an absolutely uniserial algebra, and the converse is true.

In this note we shall prove the following two theorems:

Theorem 1. *If A_L is generalized uniserial for an extension field L of K, then A has the same property.*

Theorem 2. *If A_L has the radical expressible as a principal ideal, then A has the same property.*

1. Proof of Theorem 1. Let N be the radical of A. We may assume that $N^2=0$, since A is generalized uniserial if and only if A/N^2 is generalized uniserial.[2] Let $A=\sum_i Ae_i$ be a direct decomposition of A into directly indecomposable components, and $e_i=\sum_{j=1}^{r_i} f_j^{(i)}$ be the decomposition of the idempotent e_i into primitive orthogonal idempotents in A_L. Assuming

$$r_1=\cdots=r_{n_1}\lneqq r_{n_1+1}=\cdots=r_{n_2}\lneqq\cdots\lneqq r_{n_{\lambda-1}+1}=\cdots=r_n,$$

we can classify e_κ into

$$\mathfrak{E}_1=(e_1,\dots,e_{n_1})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathfrak{E}_\lambda=(e_{n_{\lambda-1}+1},\dots,e_n).$$

Then $N_Le_i=\sum_{j=1}^{r_1} N_Lf_j^{(i)}$ ($e_i\in\mathfrak{E}_1$), where $N_Lf_j^{(i)}$ is directly indecomposable by the assumption of A_L. On the other hand, if $Ne_i\cong\sum_j\bar{A}\bar{e}_j$, where $\bar{A}\bar{e}_j=Ae_j/Ne_j$, then $N_Le_i\cong\sum_j\bar{A}_L\bar{e}_j$ and the left hand side is decomposed into r_1 directly indecomposable components. Therefore by the Remak Schmidt Theorem, the right hand side has r_1 directly indecomposable components. But from the assumption of r_1 we have $N_Le_i\cong\bar{A}_L\bar{e}_j$ and $Ne_i\cong\bar{A}\bar{e}_j$ ($e_j\in\mathfrak{E}_1$). Moreover $e_{j'}Ne_i=0$ for $e_{j'}\notin\mathfrak{E}_1$. Similarly $e_iN\cong\bar{e}_k\bar{A}$ ($e_k\in\mathfrak{E}_1$) for $e_i\in\mathfrak{E}_1$ and $e_iNe_{j'}=0$ for $e_{j'}\notin\mathfrak{E}_1$.

1) T. Yoshii [3].
2) T. Nakayama [2], Lemma 3.

Now suppose that $Ne_i \cong \sum_j \bar{A}\bar{e}_j$ $(e_i \in \mathfrak{E}_2)$. Then, by the above consideration, $\bar{A}\bar{e}_\lambda$ $(e_\lambda \in \mathfrak{E}_1)$ does not appear in the right hand side. Therefore we can use the same method as in the case of $\mathfrak{E}_1$, and so we have $Ne_i \cong \bar{A}\bar{e}_j$ for $e_j \in \mathfrak{E}_2$ and $e_{j'}Ne_i = e_iNe_{j'} = 0$ for $e_{j'} \notin \mathfrak{E}_2$. Similarly we can prove that each Ne_i (or e_iN) is simple. Therefore A is generalized uniserial.

Remark. If A_L is left (or right) generalized uniserial but is not generalized uniserial, then A is not always generalized uniserial. This is shown by the next example.

Let

$$A = Ke_1 + Ke_2 + K\omega e_2 + Ku + K\omega u \qquad (\omega^2 = k \in K).$$

	e_1	e_2	ωe_2	u	ωu
e_1	e_1	0	0	u	ωu
e_2	0	e_2	ωe_2	0	0
ωe_2	0	ωe_2	ke_2	0	0
u	0	u	ωu	0	
ωu	0	ωu	ku		

Then

$$Ae_1 = Ke_1,$$
$$Ae_2 = Ke_2 + K\omega e_2 + Ku + K\omega u,$$
$$Ne_2 = Ku + K\omega u \cong 2Ae_1.$$

Thus A is not left generalized uniserial. If we put $K(\omega) = W$, $e_2 = f_1 + f_2$ in A_W, then

$$(Ne_2)_W = N_W f_1 + N_W f_2 \cong 2A_W e_1.$$

Hence

$$N_W f_1 \cong A_W e_1, \qquad N_W f_2 \cong A_W e_1,$$

and thus A_W is a left generalized uniserial algebra.

2. Proof of Theorem 2. The radical of A is expressible as a principal ideal if and only if the following two conditions are satisfied;

i) A is generalized uniserial,

ii) A is quasi-primary-decomposable.[3]

Now if the radical of A_L is expressible as a principal ideal, then A_L is generalized uniserial and A is generalized uniserial by Theorem 1. Therefore we have only to prove that if A_L is generalized uniserial and quasi-primary-decomposable, then A is quasi-primary-decomposable.

We may assume that $N^2 = 0$, since A is quasi-primary-decomposable if and only if A/N^2 is quasi-primary-decomposable.[4] Let Ne_κ

3) K. Morita [1], Theorem 1.

4) A is generalized uniserial in this case. However, if A is not generalized uniserial it is not clear that A is quasi-primary-decomposable when and only when A/N^2 is quasi-primary-decomposable.

$\cong\bar{A}\bar{e}_\rho$, $N_L e_\kappa=\sum_{j=1}^{r_\kappa} N_L f_j^{(\kappa)}$ and $\bar{A}_L\bar{e}_\rho=\sum_{j=1}^{r_\rho}\bar{A}_L\bar{f}_j^{(\rho)}$. Then there exist k, l such that $N_L f_k^{(\kappa)}\cong\bar{A}_L\bar{f}_l^{(\rho)}$, and, by the Remak Schmidt Theorem, the number of components of $N_L e_\kappa$ operator isomorphic to $N_L f_k^{(\kappa)}$ is equal to that of those components of $\bar{A}_L\bar{e}_\rho$ operator isomorphic to $\bar{A}_L\bar{f}_l^{(\rho)}$.

Now let a_k be the number of directly indecomposable components of $A_L e_\kappa$ operator isomorphic to $A_L f_k^{(\kappa)}$, and a_l be that of those components of $A_l e_\rho$ operator isomorphic to $A_l f_l^{(\rho)}$. Now $N_L f_k^{(\kappa)}\cong N_L f_j^{(\kappa)}$ if and only if $A_L f_k^{(\kappa)}\cong A_L f_j^{(\kappa)}$ by the assumption that A_L is generalized uniserial, and $\bar{A}_L\bar{f}_l^{(\rho)}\cong\bar{A}_L\bar{f}_j^{(\rho)}$ if and only if $A_L f_l^{(\rho)}\cong A_L f_j^{(\rho)}$. Hence $a_l=a_k$.

Let the multiplicity of Ae_κ be $g(\kappa)$ and that of Ae_ρ be $g(\rho)$. Then the multiplicity of $A_L f_k^{(\kappa)}$ is $g(\kappa)a_k$ and that of $A_L f_l^{(\rho)}$ is $g(\rho)a_l$. But by the assumption on A_L, the multiplicity of $A_L f_k^{(\kappa)}$ is equal to that of $A_L f_l^{(\rho)}$. Therefore $g^{(\kappa)}a_k=g^{(\rho)}a_l$ and $g^{(\kappa)}=g^{(\rho)}$. Hence A is quasi-primary-decomposable. This completes the proof.

Remark. K. Morita proved[5] that the group ring of a finite group $\mathfrak{G}$ over an algebraically closed field Ω with characteristic p has the radical expressible as a principal ideal if and only if

(1) $\mathfrak{H}\mathfrak{P}$ is normal and

(2) $\mathfrak{P}$ is cyclic.

Herein $\mathfrak{H}$ is the largest normal subgroup of $\mathfrak{G}$ with an order prime to p, and $\mathfrak{P}$ is a p-Sylow subgroup of $\mathfrak{G}$. But we see, by Theorem 2, that the "if" part or Morita's result holds good for any field of characteristic p.

References

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[2] T. Nakayama: Note on uni-serial and generalized uni-serial rings, Proc. Imp. Acad., **16**, 285-289 (1940).

[3] T. Yoshii: Note on generalized uniserial algebras I, Osaka Math. Journ., **6**, No. 1, 103-105 (1954).

5) K. Morita [1], Theorem 8.