114. On a Certain Type of Analytic Fiber Bundles

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In a lecture at University of Chicago (cf. [1]), A. Weil developed the theory of algebraic fiber varieties. Among others, he treated fiber varieties over a non-singular algebraic curve, which have the projective straight line as fibers and the group of affine transformations as the structure group. He classified these fiber varieties in a purely algebraic way (and in the case of a universal domain of any characteristic). In this note we shall show that his second invariant admits a simple and natural interpretation, as far as complex analytic fiber bundles are concerned.

1. Let V be a compact complex analytic manifold. A fiber bundle $\mathfrak{B}$ to be considered here is defined in terms of a finite open covering $\{U_j\}$ of V, and a system of holomorphic mappings s_{jk} from $U_j \frown U_k$ into G; the group of the affine transformations of a complex affine straight line $\boldsymbol{C}$. Here the mappings s_{jk} satisfy the relation

$$(1) \qquad s_{jk} \cdot s_{kl} = s_{jl} \qquad \text{in } U_j \frown U_k \frown U_l.$$

If we write

$$s_{jk} \cdot \zeta = a_{jk}\zeta + b_{jk} \qquad \text{for } \zeta \in \boldsymbol{C},$$

then a_{jk} and b_{jk} are holomorphic functions in $U_j \frown U_k$ and

$$(2) \qquad \begin{cases} a_{jk} \cdot a_{kl} = a_{jl} \\ a_{jk} \cdot b_{kl} + b_{jk} = b_{jl}, \end{cases}$$

while $\mathfrak{B}$ may be described in terms of "coordinates" (z, ζ_j) $(z \in U_j$ and $\zeta_j \in \boldsymbol{C})$, with the relation

$$(3) \qquad (z, \zeta_j) \sim (z', \zeta_k) \text{ if and only if } \begin{cases} z = z' \in U_j \frown U_k \\ \zeta_j = a_{jk}(z)\zeta_k + b_{jk}(z). \end{cases}$$

Two systems $s_{jk} = (a_{jk}, b_{jk})$ and $s'_{jk} = (a'_{jk}, b'_{jk})$ define the same bundle if and only if

$$s'_{jk} = t_j^{-1} s_{jk} t_k,$$

where each $t_j = (c_j, d_j)$ is a holomorphic mapping of U_j into G. In terms of a, b, c and d, this condition is expressed as

$$(4) \qquad \begin{cases} a'_{jk} = c_j^{-1} \cdot a_{jk} \cdot c_k \\ b'_{jk} = c_j^{-1}(a_{jk}d_k + b_{jk} - d_j). \end{cases}$$

If $\mathfrak{B}$ is defined by (a_{jk}, b_{jk}), then (2) shows that (a_{jk}) defines a complex line bundle $\mathfrak{A}$ (abbreviation: C.L.B.) in the sense of

K. Kodaira. (Cf. [2].) Then (4) shows that $\mathfrak{A}$ is uniquely determined by $\mathfrak{B}$. $\mathfrak{A}$ shall be called, after Weil, the C.L.B. subordinate to $\mathfrak{B}$.

In order that $\mathfrak{B}$ and $\mathfrak{B}'$ should be equivalent, it is clear that the subordinate C.L.B.'s must be equivalent. Hence we assume that $\mathfrak{B}$ and $\mathfrak{B}'$ are defined by (a_{jk}, b_{jk}) and (a_{jk}, b'_{jk}) respectively, and seek for a property which distinguishes $\mathfrak{B}$ from $\mathfrak{B}'$.

The condition (4) becomes, in this case,

$$b'_{jk} = c^{-1}(a_{jk}d_k + b_{jk} - d_j), \tag{5}$$

where c is a complex constant, $\neq 0$.

2. We observe that if $\mathfrak{B}$ has a holomorphic cross section, then $\mathfrak{B}$ reduces to its subordinate C.L.B. In fact, a holomorphic cross section is determined by a system $\varphi = (\varphi_j)$ of holomorphic functions φ_j in U_j, with the property

$$\varphi_j(z) = a_{jk}(z)\varphi_k(z) + b_{jk}(z).$$

Then by a transformation of the origin

$$\zeta_j \to \zeta_j - \varphi_j(z)$$

in each fiber, we see that $\mathfrak{B}$ is reduced to $\mathfrak{A}$.

On the other hand, since the fibers are, topologically, nothing but the real Euclidean space of dimension 2, there exist always continuous (and hence differentiable) cross sections.

Take a C^∞ cross section $\alpha = (\alpha_j)$, then

$$\alpha_j(z, \bar{z}) = a_{jk}(z) \cdot \alpha_k(z, \bar{z}) + b_{jk}(z),$$

and therefore

$$d''\alpha_j = a_{jk} \cdot d''\alpha_k, \tag{6}$$

where d'' denotes the exterior differentiation with respect to $\bar{z}$. This shows that the system of C^∞-forms $\gamma = (d''\alpha_j)$ is a differential form with coefficients in $\mathfrak{A}$. (Cf. [2].)

Actually, γ is a d''-closed form, and if we take another cross section $\alpha' = (\alpha'_j)$ of $\mathfrak{B}$, then it is clear that

$$\alpha'_j = \alpha_j + \beta_j \qquad \text{with} \quad \beta_j = a_{jk} \cdot \beta_k.$$

Hence

$$\gamma' = \gamma + d''\beta.$$

This shows that the system (a_{jk}, b_{jk}) determines an element $\tilde{\gamma}$ of $H^{0,1}(\mathfrak{A})$; the d''-cohomology group of C^∞-forms on V with coefficients in $\mathfrak{A}$ and of type $(0, 1)$.

Conversely, let $\tilde{\gamma} \in H^{0,1}(\mathfrak{A})$ be given and let $\gamma = (\gamma_j)$ be a form in the class $\tilde{\gamma}$. Then

$$d''\gamma_j = 0 \qquad \text{in } U_j.$$

If we take a refinement $\{V_\lambda\}$ of the covering $\{U_j\}$, and associate to each λ an index j such that $V_\lambda \subset U_j$, then we can speak of $a_{\lambda\mu}$ or γ_λ instead of a_{jk} or γ_j. If $\{V_\lambda\}$ is sufficiently fine, we can find C^∞-functions α_λ such that

$$d''\alpha_\lambda = \gamma_\lambda \qquad \text{in } V_\lambda.$$

We put

$$b_{\lambda\mu} = \alpha_\lambda - a_{\lambda\mu}\alpha_\mu,$$

then $d''b_{\lambda\mu}=0$ and $b_{\lambda\mu}$ is holomorphic in $V_\lambda \frown V_\mu$, and it is easy to see that the system $(a_{\lambda\mu}, b_{\lambda\mu})$ satisfies (2). Hence it defines a $\mathfrak{B}$, to which $\mathfrak{A}$ is subordinate. It is also clear that the class $\tilde{\gamma}$ is the one which is determined by $\mathfrak{B}$.

Finally, if we replace (a_{jk}, b_{jk}) by another equivalent system (a_{jk}, b'_{jk}), then by (5)

$$b'_{jk} = c^{-1}(a_{jk}d_k + b_{jk} - d_j).$$

We then replace the system (α_j) of C^∞-functions by (α'_j), where

$$\alpha'_j = c^{-1}(\alpha_j - d_j),$$

then

$$\alpha'_j = a_{jk}\cdot\alpha'_k + b'_{jk}$$

and

$$d''\alpha'_j = c^{-1}d''\alpha_j.$$

Hence if we take another expression of $\mathfrak{B}$, the corresponding element in $H^{0,1}(\mathfrak{A})$ is multiplied by a non-zero constant.

The converse being true, we get

THEOREM. Let $\mathfrak{A}$ be a C.L.B. over a compact complex analytic manifold $\boldsymbol{V}$. Then the fiber bundles of type (3), to which $\mathfrak{A}$ is subordinate, are in one to one correspondence with the points of a projective space $\boldsymbol{P}$, whose representative cone is $H^{0,1}(\mathfrak{A})$ (with the only one exception of $\mathfrak{A}$ itself).

We shall call the point of $\boldsymbol{P}$ corresponding to $\mathfrak{B}$, the second invariant of $\mathfrak{B}$.

3. Now we assume that $\boldsymbol{V}$ is an algebraic variety in a projective space. Then, by a theorem of K. Kodaira and D. C. Spencer (cf. [3], [5]), a C.L.B. $\mathfrak{A}$ over $\boldsymbol{V}$ can be defined by a divisor $\boldsymbol{D}$ of $\boldsymbol{V}$. In other words, there is a divisor $\boldsymbol{D}$ and a system of local equations R_j of $\boldsymbol{D}$ in U_j, such that

$$a_{jk} = R_j/R_k. \tag{7}$$

When a bundle $\mathfrak{B}$ is given by (a_{jk}, b_{jk}), we put

$$h_{jk} = b_{jk}/R_j, \tag{8}$$

then each h_{jk} is a meromorphic function in $U_j \frown U_k$, with $(h_{jk}) + \boldsymbol{D} \succ 0$. From (2), it follows that

$$\begin{cases} h_{jk} + h_{kj} = 0, \\ h_{jk} + h_{kl} + h_{lj} = 0 \qquad \text{in } U_j \frown U_k \frown U_l. \end{cases} \tag{9}$$

To show that our second invariant of $\mathfrak{B}$ is identical with Weil's one in the case of a curve, we proceed as follows:

Consider a d''-closed $(0, 1)$-form $\gamma=(\gamma_j)$ with coefficients in $\mathfrak{A}$ and a d''-closed $(m, m-1)$-form $\omega=(\omega_j)$ with coefficients in $-\mathfrak{A}$, where m is the dimension of $\boldsymbol{V}$. We define a product $\langle\gamma, \omega\rangle$ of γ and ω by

$$(10) \qquad \langle\gamma, \omega\rangle=\int_V \gamma_j\wedge\omega_j.$$

Since $\gamma_j=a_{jk}\gamma_k$ and $\omega_j=a_{jk}^{-1}\omega_k$, this is well defined.

If γ or ω is d''-total, then we have $\langle\gamma, \omega\rangle=0$. In fact, if $\gamma_j=d''\beta_j$ with $\beta_j=a_{jk}\beta_k$, then

$$\langle\gamma, \omega\rangle=\int_V d''(\beta_j\wedge\omega_j)=\int_V d(\beta_j\wedge\omega_j)=0,$$

because $\beta_j\wedge\omega_j$ is a form on the whole $\boldsymbol{V}$.

Hence (10) defines the product of the classes of γ and ω and thus defines the product between $H^{0,1}(\mathfrak{A})$ and $H^{m,m-1}(-\mathfrak{A})$.

Actually, these two modules are in duality by the relation (10), because, by the theory of harmonic integrals, we can set up an isomorphism

$$H^{0,1}(\mathfrak{A})\ni\tilde{\gamma}\longrightarrow\tilde{\gamma}^{+}\in H^{m,m-1}(-\mathfrak{A})$$

in such a way that $\langle\tilde{\gamma}, \tilde{\gamma}^{+}\rangle>0$ for $\tilde{\gamma}\neq 0$. (Cf. [4], [5].)
Hence $H^{0,1}(\mathfrak{A})$ can be considered as the space of linear functions on $H^{m,m-1}(-\mathfrak{A})$.

Now, we assume that $\boldsymbol{V}$ is a curve Γ, U_j are open sets in Zariski topology and a_{jk} and b_{jk} are rational functions on Γ. Then h_{jk} are also rational functions and ω_j are of type $(1, 0)$. Since $d''\omega_j=0$, ω_j are holomorphic differentials in U_j and $R_j\omega_j=R_k\omega_k$ is a meromorphic differential on the whole Γ, which we denote by the letter $\bar{\omega}$. It is clear that $\bar{\omega}$ is in the space $\mathfrak{W}(-\boldsymbol{D})$ of differentials with $(\bar{\omega})+\boldsymbol{D}\succ 0$. It is also clear that $H^{0,1}(\mathfrak{A})$ and $\mathfrak{W}(-\boldsymbol{D})$ are isomorphic by $H^{0,1}(\mathfrak{A})\ni(\omega_j)\longleftrightarrow R_j\omega_j=\bar{\omega}\in\mathfrak{W}(-\boldsymbol{D})$.

Let (a_j) be a C^∞ cross section of $\mathfrak{B}$ defined by (a_{jk}, b_{jk}). Let $U_0=\Gamma-\sum_k \boldsymbol{P}_k$ be the intersection of U_j and take an open set U_k for each k, with $\boldsymbol{P}_k\in U_k$. Then

$$\int_\Gamma d''a_j\wedge\omega_j=\lim_{\varepsilon\to 0}\int_{\Gamma-\Sigma S_\varepsilon(k)} d(a_0\omega_0)$$
$$=\lim_{\varepsilon\to 0}\sum_k\int_{\partial S_\varepsilon(k)} a_0\omega_0,$$

where $S_\varepsilon(k)$ denotes a geodesic circle of radius ε, with center $\boldsymbol{P}_k$. In the neighborhood of $\boldsymbol{P}_k$, we have

$$a_0 = a_{0k} a_k + b_{0k}, \quad \omega_0 = a_{0k}^{-1} \omega_k \quad \text{and} \quad a_{0k} = R_0 / R_k,$$

therefore

$$\int_{\partial S_\varepsilon(k)} a_0 \omega_0 = \int_{\partial S_\varepsilon(k)} a_k \omega_k + \int_{\partial S_\varepsilon(k)} (b_{0k}/a_{0k}) \omega_k.$$

The first term on the right hand side gives, when ε tends to 0, the limit 0, and the second gives

$$\int_{\partial S_\varepsilon(k)} h_{0k} \cdot (R_k \omega_k) = \int_{\partial S_\varepsilon(k)} h_{0k} \bar{\omega} = 2\pi\sqrt{-1}\, \mathrm{Res}_{P_k}(h_{0k}\bar{\omega}).$$

Hence

$$\langle (d'' a_j), \omega \rangle = \int_F d'' a_j \wedge \omega_j = 2\pi\sqrt{-1} \sum_k \mathrm{Res}_{P_k}(h_{0k} \bar{\omega}),$$

this shows that our second invariant $(d''a_j)$ is the same as Weil's one.

4. Returning to the general case of any dimension m, we can express the second invariant explicitly in terms of R_j and h_{jk} (or R_j and b_{jk}).

Take a partition of unity $1 = \sum f_j$, subordinate to the covering $\{U_j\}$, then

$$\int_V d'' a_j \wedge \omega_j = \sum_j \int_V f_j \cdot (d'' a_j \wedge \omega_j)$$

$$= \sum_j \int_V d(f_j a_j \omega_j) - \sum_j \int_V d f_j \wedge a_j \omega_j = -\sum_j \int_V d'' f_j \wedge a_j \omega_j.$$

For a point $\boldsymbol{P}$ of $\boldsymbol{V}$, let $\boldsymbol{P} \in U_{j_0} \frown \cdots \frown U_{j_q}$ and $\boldsymbol{P} \notin U_j$ for $j \neq j_\mu$, then in the neighborhood of $\boldsymbol{P}$ we have

$$a_{j_\mu} = a_{j_\mu j_0} a_{j_0} + b_{j_\mu j_0}, \quad \omega_{j_\mu} = a_{j_\mu j_0}^{-1} \omega_{j_0},$$

$$\sum_j d'' f_j \wedge a_j \omega_j = \sum_\mu d'' f_{j_\mu} \wedge a_{j_0} \omega_{j_0} + \sum_\mu d'' f_{j_\mu} \wedge h_{j_\mu j_0} \bar{\omega} = \sum_j d'' f_j \wedge h_{j j_0} \bar{\omega}.$$

Now $\sum_j d'' f_j h_{j j_0}$ does not depend on j_0, and hence is a differential form on the whole $\boldsymbol{V}$ (with singularities). In fact

$$\sum_j d'' f_j \cdot h_{j j_1} - \sum_j d'' f_j h_{j j_0} = \sum_j d'' f_j h_{j_0 j_1} = 0.$$

Put

$$\gamma_j = -R_j \sum_k d'' f_k h_{k j_0},$$

then γ_j is C^∞ in U_j and $\gamma = (\gamma_j)$ is a d''-closed $(0, 1)$-form with coefficients in $\mathfrak{A}$. The above formula shows that $\langle (d'' a_j), \omega \rangle = \langle \gamma, \omega \rangle$, hence

$$\gamma \sim (d'' a_j).$$

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