113. *Note on Deformation Retract*

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1. The main object of this note is to study a mapping which has a torus as the image space. The methods of the paper are strongly influenced by Spanier's proofs [5].

2. In this section we prepare some definitions and lemmas known in Spanier's paper on Borsuk's cohomotopy groups [5], [2].

Let $\mathfrak{X}$ denote the space of a sequence of real numbers $y=(y_i)$ $(i=1,2,\ldots)$ which are finitely non-zero (i.e. $y_i=0$ except for a finite set of integers i). $\mathfrak{X}$ is metrized by

$$\text{dist}\,(y,y')=(\sum_i (y_i-y'_i)^2)^{\frac{1}{2}}.$$

Definition 2.1. The sets below are defined by the corresponding condition on the right:

$$S^n = \{y\in\mathfrak{X}\mid y_i=0 \quad \text{for} \quad i>n+1 \quad \text{and} \sum_{1\leq i\leq n+1} y_i^2=1\},$$
$$E^{n+1} = \{y\in\mathfrak{X}\mid y_i=0 \quad \text{for} \quad i>n+1 \quad \text{and} \sum_{1\leq i\leq n+1} y_i^2\leq 1\},$$
$$E^n_+ = \{y\in S^n\mid y_{n+1}\geqq 0\},$$
$$E^n_- = \{y\in S^n\mid y_{n+1}\leqq 0\},$$
$$E^0_+=p=(1,0,\ldots,0,\ldots),$$
$$E^0_- =\bar{p}=(-1,0,\ldots,0,\ldots),$$
$$T^{2n}=S^n\times S^n,\ q=p\times p,\ \bar{q}=\bar{p}\times\bar{p} \qquad (\text{for} \quad n\geqq 1).$$

Lemma 2.2. Let A be a deformation retract [4] of a compact space X and let f: $(X,A)\to(Y,B)$ be a map of (X,A) onto (Y,B), which maps $X-A$ homeomorphically onto $Y-B$. Then B is a deformation retract of Y.

Lemma 2.3. Let (X,A) be a compact pair with $\dim(X-A)\leqq n$. If F is any closed subset of $X\times I-A\times I$, $\dim F\leqq n+1$.

Definition 2.4. Let f: $(X,A)\to(Y\times Y,(y,y))$. A map

$$F:\quad (X\times I, A\times I)\to(Y\times Y,(y,y))$$

will be called a normalizing homotopy for f, if

$$\left.\begin{array}{l} F(x,0)=f(x) \\ F(x,1)\in(Y\times y)\cup(y\times Y)\end{array}\right\}\ \text{for all}\quad x\in X.$$

The map f': $(X,A)\to[(Y\times y)\cup(y\times Y),(y,y)]$ defined by $f'(x)=F(x,1)$ is called a normalization of f.

In the following $Y\vee Y$ will denote the space $(Y\times y)\cup(y\times Y)$. Let

$$\Omega:\quad [Y\vee Y,(y,y)]\to(Y,y)$$

be defined by

$$(y', y) = y' \quad \text{for} \quad (y', y) \in Y \times y,$$
$$(y, y'') = y'' \quad \text{for} \quad (y, y'') \in y \times Y.$$

Definition 2.5. Let α, β: $(X, A) \to (Y, B)$ and assume that $\alpha \times \beta$: $(X, A) \to (Y \times Y, (y, y))$ can be normalized. Let f: $(X, A) \to (Y \vee Y, (y, y))$ be a normalization of $\alpha \times \beta$. The sum with respect to f(denoted by $\alpha < f > \beta$) is defined to be the composite map $\alpha < f > \beta = \Omega f$.

Lemma 2.6. Let (X, A) be a pair with dim $F < 2n$ any closed $F \subset X - A$. If f: $(X, A) \to (S^n \times S^n, (p, p))$, there exists a normalization g of f such that $f \simeq g$ rel $f^{-1}(S^n \vee S^n)$.

Lemma 2.7. Let (X, A) be a compact pair with dim $(X - A) < 2n - 1$. If $\alpha, \beta, \alpha', \beta'$: $(X, A) \to (S^n, p)$ with $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$ and if g: $(X, A) \to (S^n \vee S^n, (p, p))$ is a normalization of $\alpha \times \beta$ and g': $(X, A) \to (S^n \vee S^n, (p, p))$ is a normalization of $\alpha' \times \beta'$, then $\Omega g \simeq \Omega g'$.

3. Our theorems are the following:

Theorem 3.1. In the product space $T^{2n} \times T^{2n}$, the subset $(T^{2n} \times q) \cup (q \times T^{2n})$ is a deformation retract of $T^{2n} \times T^{2n} - [(\bar{q}, \bar{q}) \cup T^{2n} \times \bar{p} \times p \cup T^{2n} \times p \times \bar{p} \cup \bar{p} \times p \times T^{2n} \cup p \times \bar{p} \times T^{2n}]$.

Proof. This is analogous to Borsuk's proof [1]. Let f: $(E^n, S^{n-1}) \to (S^n, p)$ map $E^n - S^{n-1}$ homeomorphically onto $S^n - p$. Let $f^{-1}(\bar{p}) = \bar{x}$ be the center of E^n. Define

$$\begin{aligned} g\colon\ & [E^{2n} \times E^{2n} - (\bar{x}, \bar{x}, \bar{x}, \bar{x}) \cup E^{2n} \times (E^n)^i \times S^{n-1} \cup E^{2n} \times S^{n-1} \\ & \times (E^n)^i \cup (E^n)^i \times S^{n-1} \times E^{2n} \cup S^{n-1} \times (E^n)^i \times E^{2n}, \\ & E^{2n} \times T^{2n-2} \cup T^{2n-2} \times E^{2n}] \\ & \to [T^{2n} \times T^{2n} - (\bar{q}, \bar{q}) \cup T^{2n} \times f((E^n)^i) \times p \cup T^{2n} \times p \times f((E^n)^i) \\ & \cup f((E^n)^i) \times p \times T^{2n} \cup p \times f((E^n)^i) \times T^{2n}, \\ & T^{2n} \times q \cup q \times T^{2n}] \end{aligned}$$

by

$g(x_1, x_2, x_3, x_4) = (f(x_1), f(x_2), f(x_3), f(x_4))$, where $(E^n)^i$ is the interior of a set E^n. Then g is a map onto $T^{2n} \times T^{2n} - (\bar{p}, \bar{p}, \bar{p}, \bar{p}) \cup T^{2n} \times f((E^n)^i) \times p \cup T^{2n} \times p \times f((E^n)^i) \cup f((E^n)^i) \times p \times T^{2n} \cup p \times f((E^n)^i) \times T^{2n}$ which maps $E^{2n} \times E^{2n} - (\bar{x}, \bar{x}, \bar{x}, \bar{x}) \cup E^{2n} \times (E^n)^i \times S^{n-1} \cup E^{2n} \times S^{n-1} \times (E^n)^i \cup (E^n)^i \times S^{n-1} \times E^{2n} \cup S^{n-1} \times (E^n)^i \times E^{2n} - [E^{2n} \times T^{2n-2} \cup T^{2n-2} \times E^{2n}]$ homeomorphically onto

$$\begin{aligned} & T^{2n} \times T^{2n} - (\bar{p}, \bar{p}, \bar{p}, \bar{p}) \cup T^{2n} \times f((E^n)^i) \times p \cup T^{2n} \times p \times f((E^n)^i) \\ & \cup f((E^n)^i) \times p \times T^{2n} \cup p \times f((E^n)^i) \times T^{2n} - [T^{2n} \times q \cup q \times T^{2n}]. \end{aligned}$$

Since $E^{2n} \times E^{2n}$ is a $4n$-cell with center $(\bar{x}, \bar{x}, \bar{x}, \bar{x})$ and the intersection of $(E^{2n} \times E^{2n})$ and $E^{2n} \times E^{2n} - [E^{2n} \times (E^n)^i \times S^{n-1} \cup E^{2n} \times S^{n-1} \times (E^n)^i \cup (E^n)^i \times S^{n-1} \times E^{2n} \cup S^{n-1} \times (E^n)^i \times E^{2n}]$ is $E^{2n} \times T^{2n-2} \cup T^{2n-2} \times E^{2n}$, it is clear that $E^{2n} \times T^{2n-2} \cup T^{2n-2} \times E^{2n}$ is a deformation retract of $E^{2n} \times E^{2n} - [(\bar{x}, \bar{x}, \bar{x}, \bar{x}) \cup E^{2n} \times (E^n)^i \times S^{n-1} \cup E^{2n} \times S^{n-1} \times (E^n)^i \cup (E^n)^i \times S^{n-1} \times E^{2n} \cup S^{n-1} \times (E^n)^i \times E^{2n}$. Therefore, by Lemma 2.2, $T^{2n} \times q \cup q \times T^{2n}$ is a deformation retract of $T^{2n} \times T^{2n} - [(\bar{q}, \bar{q}) \cup T^{2n} \times f((E^n)^i) \times p \cup T^{2n} \times p \times f((E^n)^i) \cup f((E^n)^i) \times p \times T^{2n} \cup p \times f((E^n)^i) \times T^{2n}$. $T^{2n} \times T^{2n} - T^{2n} \times \bar{p}$

$\times p$ may be deformed onto $T^{2n}\times T^{2n}-T^{2n}\times f((E^n)^i)\times p$ and the similar deformations may be used for the another parts of the above set. Therefore, $T^{2n}\times q\cup q\times T^{2n}$ is a deformation retract of $T^{2n}\times T^{2n}-[(\bar{q},\bar{q})\cup T^{2n}\times\bar{p}\times p\cup T^{2n}\times p\times\bar{p}\cup\bar{p}\times p\times T^{2n}\cup p\times\bar{p}\times T^{2n}]$.

Theorem 3.2. In the product space $T^{2n}\times T^{2n}\times T^{2n}$, the subset $(T^{2n}\times T^{2n}\times q)\cup(T^{2n}\times q\times T^{2n})\cup(q\times T^{2n}\times T^{2n})$ is a deformation retract of $T^{2n}\times T^{2n}\times T^{2n}-[(\bar{q},\bar{q},\bar{q})\cup T^{2n}\times T^{2n}\times\bar{p}\times p\cup T^{2n}\times T^{2n}\times p\times\bar{p}\cup T^{2n}\times\bar{p}\times p\times T^{2n}\cup T^{2n}\times p\times\bar{p}\times T^{2n}\cup\bar{p}\times p\times T^{2n}\times T^{2n}\cup p\times\bar{p}\times T^{2n}\times T^{2n}$. Since the proof is similar to that of Theorem 3.1, it will be omitted.

Lemma 3.3. Let (E, A) be a pair with dim $F\leqq 4n-1$ for any closed $F\subset X-A$. Given a continuous map $f\colon (X, A)\rightarrow(Y, B)$ into a pair (Y, B) and given an open $2n$-simplex σ, where $T^{2n}\times\sigma\cup\sigma\times T^{2n}$ lies on Y and its closure $T^{2n}\times\bar{\sigma}\cup\bar{\sigma}\times T^{2n}$ does not meet B, there is a map $g\colon (X, A)\rightarrow(Y, B)$ such that $f\simeq g$ rel $f^{-1}(Y-T^{2n}\times\sigma\cup\sigma\times T^{2n})$ and $g(X)\subset Y-T^{2n}\times\sigma\cup\sigma\times T^{2n}$.

Proof. Let $\sigma=\bar{\sigma}-\sigma$ be the point set boundary of σ. Let $M=f^{-1}(T^{2n}\times\bar{\sigma}\cup\bar{\sigma}\times T^{2n})$ and $N=f^{-1}(T^{2n}\times\dot{\sigma}\cup\dot{\sigma}\times T^{2n})$. Then N is a closed subset of M, and $\dim M\leqq 4n-1$. The map $f|N$, which maps N into $T^{2n}\times\sigma\cup\dot{\sigma}\times T^{2n}$, as an extension $f'\colon M\rightarrow T^{2n}\times\dot{\sigma}\cup\dot{\sigma}\times T^{2n}$, because we consider the first coordinate for $f(N)\cap[\dot{\sigma}\times T^{2n}]$ and the second coordinate for $f(M)\cap[T^{2n}\times\dot{\sigma}]$ and use the methods shown by Dowker [3]. Define

$$g\colon (X, A)\rightarrow(Y, B)$$

by

$$g(x)=\begin{cases} f'(x) & \text{if } x\in M,\\ f(x) & \text{if } x\in X-M.\end{cases}$$

Then g is continuous and $g(X)\subset Y-T^{2n}\times\sigma\cup\sigma\times T^{2n}$. Moreover, f' and $f|M$ are two maps of (M, N) into $(T^{2n}\times\bar{\sigma}\cup\bar{\sigma}\times T^{2n}, T^{2n}\times\dot{\sigma}\cup\dot{\sigma}\times T^{2n})$ which agree on N and hence are homotopic relative to N. Let

$$F\colon (M\times I, N\times I)\rightarrow(T^{2n}\times\bar{\sigma}\cup\bar{\sigma}\times T^{2n}, T^{2n}\times\dot{\sigma}\cup\dot{\sigma}\times T^{2n})$$

be a homotopy between f' and $f|M$ relative to N. Define

$$G\colon (X\times I, A\times I)\rightarrow(Y, B)$$

by

$$G(x, t)=\begin{cases} F(x, t) & \text{if } x\in M,\\ f(x) & \text{if } x\in X-M.\end{cases}$$

Then G is a homotopy rel $f^{-1}(Y-T^{2n}\times\sigma\cup\sigma\times T^{2n})$ between f and g.

Lemma 3.4. Let (X, A) be a pair with $\dim F\leqq 6n-1$ for any closed $F\subset X-A$. Given a continuous map $f\colon (X, A)\rightarrow(Y, B)$ into a pair (Y, B) and given an open $2n$-simplex σ, where $T^{2n}\times T^{2n}\times\sigma\cup T^{2n}\times\sigma\times T^{2n}\cup\sigma\times T^{2n}\times T^{2n}$ lies on Y and its closure $T^{2n}\times T^{2n}\times\bar{\sigma}\cup T^{2n}\times\bar{\sigma}\times T^{2n}\cup\bar{\sigma}\times T^{2n}\times T^{2n}$ does not meet B, there is a map $g\colon (X, A)\rightarrow$

(Y, B) such that $f \simeq g$ rel $f^{-1}(Y - T^{2n} \times T^{2n} \times \sigma \cup T^{2n} \times \sigma \times T^{2n} \cup \sigma \times T^{2n} \times T^{2n})$ and $g(X) \subset Y - T^{2n} \times T^{2n} \times \sigma \cup T^{2n} \times \sigma \times T^{2n} \cup \sigma \times T^{2n} \times T^{2n}$
Since the proof is similar to that of Lemma 3.3, it will be omitted.

Theorem 3.5. Let (X, A) be a pair with $\dim F < 4n$ for any closed $F \subset X - A$. If $f: (X, A) \to (T^{2n} \times T^{2n}, q \times q)$, there exists a normalization g of f such that $f \simeq g$ rel $f^{-1}(T^{2n} \vee T^{2n})$.

Proof. Consider $(T^{2n} \times T^{2n}, (q, q))$ as a simplicial pair subdivided in such a way that (q, q) is a vertex and $(\bar{q}, \bar{q}) \cup T^{2n} \times \bar{p} \times p \cup T^{2n} \times p \times \bar{p} \cup \bar{p} \times p \times T^{2n} \cup p \times \bar{p} \times T^{2n}$ is interior to $T^{2n} \times \sigma \cup \sigma \times T^{2n}$ whose closure $T^{2n} \times \bar{\sigma} \cup \bar{\sigma} \times T^{2n}$ does not meet $T^{2n} \vee T^{2n}$. By Lemma 3.3, there is a map $h: (X, A) \to (T^{2n} \times T^{2n}; (q, q))$ such that $h(X) \subset T^{2n} \times T^{2n} - T^{2n} \times \sigma \cup \sigma \times T^{2n} \subset T^{2n} \times T^{2n} - [(\bar{q}, \bar{q}) \cup T^{2n} \times \bar{p} \times p \cup T^{2n} \times p \times \bar{p} \cup \bar{p} \times p \times T^{2n} \cup p \times \bar{p} \times T^{2n}$ and $f \simeq h$ rel $f^{-1}(T^{2n} \times T^{2n} - T^{2n} \times \sigma \cup \sigma \times T^{2n})$. By Theorem 3.1, $T^{2n} \vee T^{2n}$ is a deformation retract of $T^{2n} \times T^{2n} - [(\bar{q}, \bar{q}) \cup T^{2n} \times \bar{p} \times p \cup T^{2n} \times p \times \bar{p} \cup \bar{p} \times p \times T^{2n} \cup p \times \bar{p} \times T^{2n}$, so there is a retracting deformation F of it onto $T^{2n} \vee T^{2n}$. Then Fh is a normalizing homotopy for h and if g is the resulting normalization of h, $g \simeq h$ rel $h^{-1}(T^{2n} \vee T^{2n}) = f^{-1}(T^{2n} \vee T^{2n})$. Therefore, $g \simeq f$ rel $f^{-1}(T^{2n} \vee T^{2n})$.

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