# 112. On the Mass Distribution Generated by a Function of P. L. Class

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§1. **Introduction.** Let $f(x, y)$ be a subharmonic function in a planar region $G$, and $\mu(e)$ be the completely additive, non-negative Borel set function generated by $f(x, y)$. Let $c(x, y; r)$ be the circle of radius $r$ with center $(x, y)$ included in the region $G$ with its boundary.

We shall introduce the functions:

$$A(f; x, y; r)=\frac{1}{\pi r^2}\int_0^{2\pi}\int_0^r f(x+\rho\cos\theta, y+\rho\sin\theta)\rho d\rho d\theta,$$

$$I(f; x, y; r)=\frac{1}{2\pi}\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)d\theta.$$

Saks[1] proved the following important theorem:

**Theorem A.** *If $f(x, y)$ is subharmonic in the region $G$, then, for almost all points $(x, y)$ in $G$, we have*

$$\lim_{r\to 0}\frac{8}{r^2}[A(f; x, y; r)-f(x, y)]=D_s\mu(x, y),$$

$$\lim_{r\to 0}\frac{4}{r^2}[I(f; x, y; r)-f(x, y)]=D_s\mu(x, y),$$

*where $D_s\mu(x, y)$ denotes the symmetric derivative of $\mu(e)$ at $(x, y)$, that is to say,*

$$D_s\mu(x, y)=\lim_{\rho\to 0}\frac{\mu[C(x, y; \rho)]}{\pi\rho^2},$$

*$C(x, y; \rho)$ being the circle completely included in $G$.*
Recently M. D. Reade[2] proved the following

**Theorem B.** *If $f(x, y)$ is a function of P. L. class in $G$, then, for almost all points $(x, y)$ in $G$, we have*

$$\lim_{r\to 0}\frac{4}{r^2}[I^2(f; x, y; r)-A(f^2; x, y; r)]=f^2(x, y)D_s\sigma(x, y),$$

*where $\sigma(e)$ denotes the mass distribution generated by $\log f(x, y)$.*
In this paper, we shall generalize this. We shall prove in §2 some lemmas and in §3 our main theorem.

§2. We prove some lemmas which will be used in §3.

**Lemma 1.** *Let $p(x, y)$, $q(x, y)$ and $p(x, y)q(x, y)$ be subharmonic in $G$, and put*

$$A(pq; x, y; r)=\frac{1}{\pi r^2}\int_0^{2\pi}\int_0^r p(x+\rho\cos\theta, y+\rho\sin\theta)q(x+\rho\cos\theta, y+\rho\sin\theta)\rho d\rho d\theta.$$

*Further, let $\mu_p(e)$, $\mu_q(e)$, $\mu_{pq}(e)$ be the mass distributions generated by $p(x, y)$, $q(x, y)$ and $p(x, y)q(x, y)$ respectively.*
*Then we have*

$$(1)\qquad \lim_{r\to 0}[L(p; x, y; r)L(q; x, y; r)-A(pq; x, y; r)]/r^2 =\frac{q(x, y)D_s\mu_p(x, y)+p(x, y)D_s\mu_q(x, y)}{4}-\frac{1}{8}D_s\mu_{pq}(x, y),$$

a.e. in $G$.

**Proof.** By the definitions we have

$$L(p)L(q)-A(pq)=\frac{1}{2}\{[L(p)+p(x, y)][L(q)-q(x, y)] +[L(p)-p(x, y)][L(q)+q(x, y)]\}-\{A(pq)-pq\},$$

and then

$$(2)\qquad \frac{L(p)L(q)-A(pq)}{r^2}=\frac{1}{8}[L(p)+p(x, y)]\frac{4}{r^2}[L(q)-q(x, y)] +\frac{1}{8}[L(q)+q(x, y)]\frac{4}{r^2}[L(p)-p(x, y)] -\frac{1}{8}\frac{8}{r^2}[A(pq)-p(x, y)q(x, y)].$$

By Theorem A, when $r\to 0$,

$$(3)\qquad \left.\begin{array}{ll}\dfrac{4[L(q)-q(x, y)]}{r^2}\to D_s\mu_q(x, y), & \dfrac{4[L(p)-p(x, y)]}{r^2}\to D_s\mu_p(x, y)\\ \dfrac{8}{r^2}[A(pq)-pq]\to D_s\mu_{pq}(x, y) & \end{array}\right\},$$

a.e. in $G$.

Since $p(x, y)$, $q(x, y)$ are subharmonic, $L(p)$ and $L(q)$ converge to $p(x, y)$ and $q(x, y)$ respectively a.e., as $r\to 0$. Therefore by (2) and (3) wet get

$$\lim_{r\to 0}\frac{L(p)L(q)-A(pq)}{r^2}=\frac{1}{4}\{p(x, y)D_s\mu_q(x, y)+q(x, y)D_s\mu_p(x, y)\} -\frac{1}{8}D_s\mu_{pq}(x, y),\qquad \text{a.e. in } G,$$

which is the required.

**Lemma 2.** *If $p(x, y)$, $q(x, y)$ and $p(x, y)\,q(x, y)$ are subharmonic in $G$, and if $e$ is a Borel set completely included in $G$, then we have*

$$\mu_{pq}(e)=\iint_e p(x,y)d\mu_q(e_p)+\iint_e q(x,y)d\mu_p(e_p)$$
$$+2\iint_e\left(\frac{\partial p}{\partial x}\frac{\partial q}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial q}{\partial y}\right)dxdy.$$

**Proof.** Let $D$ be an arbitrary domain such that $\overline{D}\subset G$.

Evans[3] and Riesz[4] proved the following facts: If we put

$$A_2(p;x,y;r)\equiv A(A(p);x,y;r),\quad A_3(p;x,y;r)\equiv A(A_2(p);x,y;r),$$
$$A_2(q;x,y;r)\equiv A(A(q);x,y;r),\quad A_3(q;x,y;r)\equiv A(A_2(q);x,y;r),$$

then there exists a positive decreasing sequence $\{\rho_n\}$ $(\rho\downarrow 0)$ such that the three sequences

$$(4)\qquad \mu_p^{(n)}(e)=\iint_e \Delta A_3(p;x,y;\rho_n)dxdy,\quad \mu_q^{(n)}(e)=\iint_e \Delta A_3(q;x,y;\rho_n)dxdy$$

and

$$(5)\qquad \mu_{pq}^{(n)}(e)=\iint_e \Delta[A_3(p;x,y;\rho_n)\times A_3(q;x,y;\rho_n)]dxdy$$

converge to $\mu_p(e)$, $\mu_q(e)$, $\mu_{pq}(e)$, respectively as $n\to\infty$, where $e$ denotes an open set $e(\bar{e}\subset G)$ and is $\mu_p$-, $\mu_q$-, $\mu_{pq}$- regular.

Now let $R$ denote orientated, $\mu_p$-, $\mu_q$-, $\mu_{pq}$- regular rectangle contained in $D$ and put $A_3(p;x,y;\rho_n)\equiv\mathfrak{A}^{(n)}(x,y)$, $A_3(q;x,y;r)\equiv\mathfrak{B}^{(n)}(x,y)$. Let us now estimate $\Delta(\mathfrak{A}^{(n)}\mathfrak{B}^{(n)})$. Since

$$\frac{\partial^2\mathfrak{A}^{(n)}\mathfrak{B}^{(n)}}{\partial x^2}=\mathfrak{A}_{xx}^{(n)}\mathfrak{B}^{(n)}+2\mathfrak{A}_x^{(n)}\mathfrak{B}_x^{(n)}+\mathfrak{A}_{xx}^{(n)}\mathfrak{B}^{(n)},$$
$$\frac{\partial^2\mathfrak{A}^{(n)}\mathfrak{B}^{(n)}}{\partial y^2}=\mathfrak{A}^{(n)}\mathfrak{B}_{yy}^{(n)}+2\mathfrak{A}_y^{(n)}\mathfrak{B}_y^{(n)}+\mathfrak{A}_{yy}^{(n)}\mathfrak{B}^{(n)},$$

we have

$$(6)\qquad \begin{aligned}\Delta(\mathfrak{A}^{(n)}\mathfrak{B}^{(n)})&=\mathfrak{A}^{(n)}(\mathfrak{B}_{xx}^{(n)}+\mathfrak{B}_{yy}^{(n)})+\mathfrak{B}^{(n)}(\mathfrak{A}_{xx}^{(n)}+\mathfrak{A}_{yy}^{(n)})+2(\mathfrak{A}_x^{(n)}\mathfrak{B}_x^{(n)}+\mathfrak{A}_y^{(n)}\mathfrak{B}_y^{(n)})\\&=\mathfrak{A}^{(n)}\Delta\mathfrak{B}^{(n)}+2(\mathfrak{A}_x^{(n)}\mathfrak{B}_x^{(n)}+\mathfrak{A}_y^{(n)}\mathfrak{B}_y^{(n)})+\mathfrak{B}^{(n)}\Delta\mathfrak{A}^{(n)}.\end{aligned}$$

By (4), (5), (6), we obtain

$$\mu_{pq}(R)=\lim_{n\to\infty}\iint_R \mathfrak{A}^{(n)}\Delta\mathfrak{B}^{(n)}dxdy+\lim_{n\to\infty}\iint_R \mathfrak{B}^{(n)}\Delta\mathfrak{A}^{(n)}dxdy$$
$$+2\lim_{n\to\infty}\iint_R(\mathfrak{A}_x^{(n)}\mathfrak{B}_x^{(n)}+\mathfrak{A}_y^{(n)}\mathfrak{B}_y^{(n)})dxdy,$$

and by (4) and (5),

$$(7)\qquad \mu_{pq}(R)=\lim_{n\to\infty}\iint_R \mathfrak{A}^{(n)}d\mu_q^{(n)}(e_p)+\lim_{n\to\infty}\iint_R \mathfrak{B}^{(n)}d\mu_p^{(n)}(e_p)$$
$$+2\lim_{n\to\infty}\iint_R(\mathfrak{A}_x^{(n)}\mathfrak{B}_x^{(n)}+\mathfrak{A}_y^{(n)}\mathfrak{B}_y^{(n)})dxdy.$$

After Frostman,[5] we have

$$(8)\qquad \lim_{n\to\infty}\iint_R \mathfrak{A}^{(n)} d\mu_q^{(n)}(e_p)=\iint_R p(x,y)d\mu_q(e_p),$$

$$\lim_{n\to\infty}\iint_R \mathfrak{B}^{(n)} d\mu_p^{(n)}(e_p)=\iint_R q(x,y)d\mu_p(e_p),$$

and after Evans[3]

$$(9)\qquad \lim_{n\to\infty}\iint_R\left[\frac{\partial\mathfrak{A}^{(n)}}{\partial x}\frac{\partial\mathfrak{B}^{(n)}}{\partial x}+\frac{\partial\mathfrak{A}^{(n)}}{\partial y}\frac{\partial\mathfrak{B}^{(n)}}{\partial y}\right]dxdy$$
$$=\iint_R\left(\frac{\partial p}{\partial x}\frac{\partial q}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial q}{\partial y}\right)dxdy.$$

By (7), (8), (9), we get the following relation.

$$(10)\qquad \mu_{pq}(R)=\iint_R p(x,y)d\mu_q(e_p)+\iint_R q(x,y)d\mu_p(e_p)$$
$$+2\iint_R\left(\frac{\partial p}{\partial x}\frac{\partial q}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial q}{\partial y}\right)dxdy.$$

By the reasoning of Reade,[2] we can see that this relation holds good for any open orientated rectangle contained in $D$.

Hence the relation (10) holds good for any Borel set contained in $D$. Since $D$ is an arbitrary domain ($\bar{D}\subset G$), Lemma 2 is completely proved.

§3. We can show now the main theorem:

**Theorem 1.** *If $p(x,y)$, $q(x,y)$ and $p(x,y)\,q(x,y)$ are subharmonic in a domain $G$, and if $\mu_p(e)$, $\mu_q(e)$ are mass distributions generated by $p(x,y)$, $q(x,y)$ respectively, then we have*

$$\lim_{r\to 0}\frac{4}{r^2}\left[L(p;x,y;r)L(q;x,y;r)-A(pq;x,y;r)\right]$$
$$=\frac{1}{2}\left[pd_s\mu_q+qd_s\mu_p\right]-\left(\frac{\partial p}{\partial x}\frac{\partial q}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial q}{\partial y}\right),\qquad \text{a.e. in } G.$$

**Proof.** By Lemma 1, we get

$$\lim_{r\to 0}\frac{4}{r^2}\left[L(p;x,y;r)L(q;x,y;r)-A(pq;x,y;r)\right]$$
$$=pD_s\mu_q+qD_s\mu_q-\tfrac{1}{2}D_s\mu_{pq},\qquad \text{a.e. in } G,$$

and by Lemma 2

$$D_s\mu_{pq}=qD_s\mu_p+pD_s\mu_q+2\left(\frac{\partial p}{\partial x}\frac{\partial q}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial q}{\partial y}\right),\qquad \text{a.e. in } G.$$

Therefore the required result is immediately obtained.

§4. We shall assume that $p(x,y)$, $q(x,y)$ are functions of P. L. class in $G$. Then $u(x,y)=\log p(x,y)$ and

$v(x, y)=\log q(x, y)$ are subharmonic in $G$. Let $\sigma_p(e)$ and $\sigma_q(e)$ denote the mass distributions generated by $u(x, y)$ and $v(x, y)$ respectively.

By a theorem of Beckenbach, if $p(x, y)$, $q(x, y)$ are functions of P. L. class, then we have for any circle $C(x, y; r)$ completely included in $G$,

$$A(pq; x, y; r) \leqq L(p; x, y; r)L(q; x, y; r).$$

We shall discuss the value of

$$\lim_{r\to 0} \frac{4}{r^2}[L(p; x, y; r)L(q; x, y; r)-A(pq; x, y; r)]$$

in terms of $\sigma_p(e)$ and $\sigma_q(e)$. For this purpose we need a lemma which was proved by M. D. Reade.[2)]

**Lemma 3.** *If $e$ is a Borel set ($\bar{e}\subset R$) and $p(x, y)$ is a function of P. L. class ($\log p(x, y)=u(x, y)$), then we have*

$$\mu_p(e)=\iint_e \exp u(x, y)d\sigma(e_p)+\iint_e \left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]p(x, y)dxdy.$$

Therefore

$$D_s\mu_p(x, y)=p(x, y)D_s\sigma_p(x, y)+p\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]. \tag{11}$$

Similarly

$$D_s\mu_q(x, y)=q(x, y)D_s\sigma_q(x, y)+q\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]. \tag{12}$$

Hence we have the following

**Theorem 2.** *If $p(x, y)$, $q(x, y)$ are positive functions of P.L. class in $G$ and if $\sigma_p(e)$, $\sigma_q(e)$ are mass distributions generated by $\log p(x, y)$ and $\log q(x, y)$, respectively, then we have*

$$\lim_{r\to 0} \frac{4}{r^2}[L(p; x, y; r)L(q; x, y; r)-A(pq; x, y; r)]=\frac{1}{2}pq\{D_s\sigma_p(x, y) + D_s\sigma_q(x, y)\}+\frac{1}{pq}\left\{\left(q\frac{\partial p}{\partial x}-p\frac{\partial q}{\partial x}\right)^2+\left(q\frac{\partial p}{\partial y}-p\frac{\partial q}{\partial y}\right)^2\right\}, \quad \text{a.e. in } G.$$

Proof. By Theorem I, we get

$$\begin{aligned} P&=\lim_{r\to 0} \frac{4}{r^2}[L(p; x, y; r)L(q; x, y; r)-A(pq; x, y; r)] \\ &=\frac{1}{2}[p(x, y)D_s\mu_q(x, y)+q(x, y)D\mu_p(x, y)] \\ &\qquad -\left(\frac{\partial p}{\partial x}\frac{\partial q}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial q}{\partial y}\right), \quad \text{a.e. in } G. \end{aligned} \tag{13}$$

(11) and (12) give us

$$P=\frac{1}{2}\Big[p\Big\{qD_s\sigma_q+q\Big(\frac{\partial\log q}{\partial x}\Big)^2+q\Big(\frac{\partial\log q}{\partial y}\Big)^2\Big\}$$

$$+q\Big\{pD_s\sigma_p+p\Big(\frac{\partial\log p}{\partial x}\Big)^2+q\Big(\frac{\partial\log p}{\partial y}\Big)^2\Big\}\Big]-\Big(\frac{\partial p}{\partial x}\frac{\partial q}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial q}{\partial y}\Big)$$

$$=\frac{1}{2}\Big[pq(D_s\sigma_p+D_s\sigma_q)+\frac{q}{p}\Big\{\Big(\frac{\partial p}{\partial x}\Big)^2+\Big(\frac{\partial p}{\partial y}\Big)^2\Big\}+\frac{p}{q}\Big\{\Big(\frac{\partial q}{\partial y}\Big)^2+\Big(\frac{\partial q}{\partial y}\Big)^2\Big\}\Big]$$

$$-\Big(\frac{\partial p}{\partial x}\frac{\partial q}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial q}{\partial y}\Big)$$

$$=\frac{1}{2}pq(D_s\sigma_p+D_s\sigma_q)+\frac{1}{pq}\Big\{\Big(q\frac{\partial p}{\partial x}-p\frac{\partial q}{\partial x}\Big)^2+\Big(q\frac{\partial p}{\partial y}-p\frac{\partial q}{\partial y}\Big)^2\Big\},$$

which is the required.

## References

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