111. Uniform Convergence of Fourier Series

By Masako SATÔ
Mathematical Institute, Tokyo Metropolitan University

(Comm. by Z. SUETUNA, M.J.A., July 12, 1954)

J. P. Nash[1)] has proved the following theorem.

Theorem 1. *If $f(x)$ is of class $\phi(n)$ with bounded $\phi'(n)$ and is continuous with modulus of continuity $\omega(\delta)$, then there exist positive constants A, B and C independent of $f(x)$ such that*

$$|s_n(x)-f(x)| \leqq \omega\left(\frac{1}{n}\right)\left[A\log\theta(n)+B\frac{n}{\phi(n)}\right]+\frac{C}{\theta(n)},$$

where $\theta(n)$ is monotone increasing and

$$1 \leqq \theta(n) \leqq \phi(n);\quad 1 \leqq \frac{\theta(n+1)}{\theta(n)} \leqq \frac{\phi(n+1)}{\phi(n)}.$$

In this theorem, a function $f(x)$ is said to be of class $\phi(n)$ if

$$\phi(n)\int_a^b f(x+t)\cos nt\,dt = O(1)$$

uniformly for all x, n, a, b with $b-a \leqq 2\pi$.

We shall prove the following generalization which contains the Dini-Lipschitz test as a particular case.

Theorem 2. *If $f(x)$ is of class $\phi(n)$, $\phi(n)$ being $O(n)$,[2)] and is continuous with modulus of continuity $\omega(\delta)$, then there exist positive constants A, B and C independent of $f(x)$ such that*

$$(1)\qquad |s_n(x)-f(x)| \leqq \omega\left(\frac{1}{n}\right)\left[A\log\theta(n)+B\log\frac{n}{\phi(n)}\right]+\frac{C}{\theta(n)},$$

where $\theta(n)$ is monotone increasing and $1 \leqq \theta(n) \leqq \phi(n)$.

Proof. It is sufficient to prove (1) for

$$s_n^*(x)-f(x)=\frac{1}{\pi}\int_0^\pi [f(x+t)+f(x-t)-2f(x)]\frac{\sin nt}{2\tan t/2}dt.$$

We divide the integral into three parts such that

$$s_n^*(x)-f(x)=\frac{1}{\pi}\left[\int_0^{\alpha/\phi(n)}+\int_{\alpha/\phi(n)}^{\beta\theta(n)/\phi(n)}+\int_{\beta\theta(n)/\phi(n)}^{\pi}\right]$$
$$=\frac{1}{\pi}[I+J+K]$$

1) J. P. Nash: Uniform convergence of Fourier series, The Rice Institute Pamphlet (1953).

In this paper we use the notation in Zygmund, Trigonometrical series, 1936.

2) As J. P. Nash shows, the assumption $\phi(n)=O(n)$ does not loose generality.

say, where α and β are the least numbers $\geqq 1$ such that $\alpha n/\pi\phi(n)$ and $\beta n\theta(n)/\pi\phi(n)$ are odd integers. Then[3]

$$I=\int_0^{\alpha/\phi(n)}[f(x+t)+f(x-t)-2f(x)]\frac{\sin nt}{2\tan t/2}dt$$

$$=\sum_{k=0}^{\{\alpha n/\pi\phi(n)\}-1}\int_{k\pi/n}^{(k+1)\pi/n}[f(x+t)+f(x-t)-2f(x)]\frac{\sin nt}{2\tan t/2}dt$$

$$=\sum_{k=1}^{\alpha\gamma}\int_{\pi/n}^{2\pi/n}(-1)^{k-1}\Big[f\Big(x+t+\frac{k-1}{n}\pi\Big)+f\Big(x-t-\frac{k-1}{n}\pi\Big)-2f(x)\Big]\cdot\frac{\sin nt}{2\tan t/2}dt$$

$$=\sum_{k=0}^{(\alpha\gamma-3)/2}\int_{\pi/n}^{2\pi/n}\Big[\Big\{f\Big(x+t+\frac{2k}{n}\pi\Big)-f\Big(x+t+\frac{2k+1}{n}\pi\Big)\Big\}-\Big\{f\Big(x-t-\frac{2k}{n}\pi\Big)-f\Big(x-t-\frac{2k+1}{n}\pi\Big)\Big\}\Big]\frac{\sin nt}{2\tan(t+2k\pi/n)/2}dt$$

$$+\sum_{k=0}^{(\alpha\gamma-3)/2}\int_{\pi/n}^{2\pi/n}\Big[\Big\{f\Big(x+t+\frac{2k+1}{n}\pi\Big)-f(x)\Big\}+\Big\{f\Big(x-t-\frac{2k+1}{n}\pi\Big)-f(x)\Big\}\Big]\cdot\Big[\frac{1}{2\tan(t+2k\pi/n)/2}-\frac{1}{2\tan(t+(2k+1)\pi/n)/2}\Big]\sin nt\,dt$$

$$=I_1+I_2$$

say, where for the sake of brevity we put $\gamma=n/\pi\phi(n)$. We have

$$|I_1|\leqq\sum_{k=0}^{(\alpha\gamma-3)/2}\int_{\pi/n}^{2\pi/n}\Big[\frac{|f(x+t+2k\pi/n)-f(x+t+(2k+1)\pi/n)|}{t+2k\pi/n}+\frac{|f(x-t-2k\pi/n)-f(x-t-(2k+1)\pi/n)|}{t+2k\pi/n}\Big]dt$$

$$\leqq 2\omega\Big(\frac{\pi}{n}\Big)\sum_{k=0}^{(\alpha\gamma-3)/2}\frac{1}{(2k+1)\pi/n}\int_{\pi/n}^{2\pi/n}dt\leqq 2\omega\Big(\frac{\pi}{n}\Big)\sum_{k=1}^{\alpha\gamma-2}\frac{1}{k/n}\frac{1}{n}$$

$$\leqq 2\omega\Big(\frac{\pi}{n}\Big)\log\frac{\alpha n}{\pi\phi(n)}\leqq B_1\omega\Big(\frac{1}{n}\Big)\log\frac{n}{\phi(n)};$$

$$|I_2|\leqq\frac{\pi}{n}\sum_{k=0}^{(\alpha\gamma-3)/2}\int_{\pi/n}^{2\pi/n}\Big[\frac{|f(x+t+(2k+1)\pi/n)-f(x)|}{(t+2k\pi/n)^2}+\frac{|f(x-t-(2k+1)\pi/n)-f(x)|}{(t+2k\pi/n)^2}\Big]dt$$

$$\leqq\frac{\pi}{n}\sum_{k=0}^{(\alpha\gamma-3)/2}\omega\Big(\frac{2k+3}{n}\pi\Big)\int_{\pi/n}^{2\pi/n}\frac{dt}{(t+2k\pi/n)^2}\leqq\frac{B_2}{n^2}\sum_{k=1}^{\alpha\gamma}\frac{\omega(k/n)}{(k/n)^2}$$

$$=B_2\omega(1/n)\sum_{k=1}^{\alpha\gamma}\frac{1}{k}\leqq B_3\omega(1/n)\log\frac{n}{\phi(n)}.^{4)}$$

---

3) Cf. R. Salem: Comptes Rendus, **207**, 662 (1938).

4) B_1, B_2 and B_3 are absolute constants.

Thus we have

$$|I| \leqq B\omega(1/n) \log \frac{n}{\phi(n)},$$

where B is a positive constant independent of $f(x)$. Further

$$J = \int_{\alpha/\phi(n)}^{\beta\theta(n)/\phi(n)} [f(x+t)+f(x-t)-2f(x)] \frac{\sin nt}{2\tan t/2} dt$$

$$= \sum_{k=\alpha n/\pi\phi(n)}^{\{\beta n\theta(n)/\pi\phi(n)\}-1} \int_{k\pi/n}^{(k+1)\pi/n} [f(x+t)+f(x-t)-2f(x)] \frac{\sin nt}{2\tan t/2} dt$$

$$= \int_{\pi/n}^{2\pi/n} \sum_{k=\alpha\gamma-1}^{\beta\gamma\theta(n)-2} (-1)^k [f(x+t+k\pi/n)+f(x-t-k\pi/n)-2f(x)] \cdot \frac{\sin nt}{2\tan(t+k\pi/n)/2} dt.$$

By using the first mean value theorem, we have

$$J = -\frac{2}{n} \sum_{k=\alpha\gamma-1}^{\beta\gamma\theta(n)-2} \frac{(-1)^k}{2\tan(\xi+k\pi/n)/2} [f(x+k\pi/n+\xi) + f(x-k\pi/n-\xi)-2f(x)],$$

where $\pi/n \leqq \xi \leqq 2\pi/n$. Hence

$$|J| \leqq \frac{2}{\pi} \sum_{k=\alpha\gamma-1}^{\beta\gamma\theta(n)-2} \frac{1}{2k+1} [|f(x+\xi+2k\pi/n)-f(x+\xi+(2k+1)\pi/n)| + |f(x-\xi-2k\pi/n)-f(x-\xi-(2k-1)\pi/n)|]$$

$$\leqq A\omega(1/n)\log\theta(n),$$

where A is a positive constant independent of $f(x)$.

We next prove that $|K| \leqq C/\phi(n)$. By the second mean value theorem

$$|K| = \left| \int_{\beta\theta(n)/\phi(n)}^{\pi} [f(x+t)+f(x-t)-2f(x)] \frac{\sin nt}{2\tan t/2} dt \right|$$

$$\leqq \frac{\phi(n)}{\beta\theta(n)} \left| \int_{\beta\theta(n)/\phi(n)}^{\eta} [f(x+t)+f(x-t)-2f(x)] \sin nt \, dt \right|,$$

where $\beta\theta(n)/\phi(n) \leqq \eta \leqq \pi$. Since $\left| \int_a^b f(x+t) \sin nt \, dt \right| \leqq C_1/\phi(n)$ for an absolute constant C_1,

$$|K| \leqq \frac{\phi(n)}{\beta\theta(n)} \frac{4C_1}{\phi(n)} = \frac{4C_1}{\beta\theta(n)}.$$

Thus there are positive constants A, B and C such that

$$|s^*(x)-f(x)| \leqq \omega(1/n)\left[A\log\theta(n) + B\log\frac{n}{\phi(n)}\right] + \frac{C}{\theta(n)},$$

and the theorem is therefore proved.

Theorem 3. *If $f(x)$ is of class $\phi(n)$, $\phi(n)$ being $O(n)$, and is continuous with modulus of continuity $\omega(\delta)$, where $\phi(n)$ satisfies $\omega(1/n)\log n/\phi(n)\to 0$ as $n\to\infty$, and moreover if $\theta(n)$ is monotone increasing to infinity, such that $1\leqq\theta(n)\leqq\phi(n)$ and $\omega(1/n)\log\theta(n)\to 0$ as $n\to\infty$, then the Fourier series of $f(x)$ converges uniformly to $f(x)$.*

We may easily prove from Theorem 2

Corollary 1 (*Dini-Lipschitz*). *If $f(x)$ is continuous and*

$$\omega(1/n)=o(1/\log n),$$

then the Fourier series of $f(x)$ converges uniformly.

For the proof it is sufficient to take $\phi(n)=\theta(n)=\log n$ in Theorem 3.

Corollary 2. *If $f(x)$ is continuous and*

$$\omega(1/n)=o(1/\log\log n),$$

$$\int_a^b f(x+t)\cos nt\,dt=O(\log n/n),$$

uniformly in x, n, a, b, where $b-a\leqq 2\pi$, then the Fourier series of $f(x)$ converges uniformly.

For the proof it is sufficient to take $\phi(n)=n/\log n$ and $\theta(n)=\log n$ in Theorem 3.

Finally I wish to express my gratitude to Professor S. Izumi for his suggestions and encouragement.