157. The Divergence of Interpolations. I

By Tetsujiro KAKEHASHI

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The convergence of interpolation polynomials to a given function in the points which satisfy a certain condition has been studied sufficiently by Walsh and others.

Let $f(z)$ be a function which is single valued and analytic throughout the interior of the circle $C_R: |z|=R>0$ and which has singularities on C_R. Let $W_n(z)$ be a sequence of polynomials of respective degrees n such that the sequence of $W_n(z)/z^n$ converges to $\lambda(z)$ analytic and non-vanishing exterior to a circle $C_R: |z|=R'<R$ and uniformly on any closed limited point set exterior to $C_{R'}$. Then the sequence of polynomials $S_n(z; f)$ of respective degrees n found by interpolation to $f(z)$ in all the zeros of $W_{n+1}(z)$ converges to $f(z)$ uniformly on any closed set interior to C_R.

But the divergence of $S_n(z; f)$ at every point exterior to C_R is not yet established in general, as far as I know. If we choose a certain condition of $W_n(z)$ which is stronger than the condition above-mentioned, the divergence at every point exterior to C_R can be proved. (Cf. T. Kakehashi: On the convergence-region of interpolation polynomials, Journal of the Mathematical Society of Japan, **6**(1954).)

The purpose of this paper is to study the divergence of $S_n(z; f)$ which interpolate to $f(z)$ with singularities of a certain type on C_R, in the points which satisfy the condition mentioned formerly.

1. Let $\varphi(t)$ be the function single valued and analytic on the circle C_R: $|t|=R>0$, a be a point on C_R and m be a complex number. If the real part of m is positive, the integral

$$\int_{C_R} \varphi(t)\,(t-a)^{m-1}\,dt \quad ; \quad a=Re^{i\alpha}$$

exists. But if the real part of m is not positive, the above integral does not exist. For such cases, we define the finite part of the integral as follows:

$$\text{(1)} \qquad \text{Pf.} \int_{C_R} \varphi(t)\,(t-a)^{m-1}dt = \int_{C_R} \psi(t)\,(t-a)^{m+p}\,dt,$$

where $\psi(t)$ is the function single valued and analytic defined by

$$\text{(2)} \qquad \varphi(t) = \sum_{k=0}^{p} \frac{\varphi^{(k)}(a)}{k!}(t-a)^k + (t-a)^{p+1}\psi(t)$$

and p is the largest positive integer such that the real part of $m+p$ is zero or negative.

Now we can consider the following.

Lemma 1. *Let $\varphi(t)$ be the function single valued and analytic on and between the two circles C_R and $C_{R'}$: $|t|=R'<R$, and $a=Re^{i\alpha}$. Then*

$$\text{Pf.}\int_{C_R}\varphi(t)\,(t-a)^{m-1}\,dt=\int_{C_{R'}}\varphi(t)\,(t-a)^{m-1}\,dt. \tag{3}$$

From the definition of Pf. $\int_{C_R}\varphi(t)\,(t-a)^{m-1}\,dt$, we can easily verify the following equations.

$$\begin{aligned}\text{Pf.}\int_{C_R}\varphi(t)\,(t-a)^{m-1}\,dt&=\int_{C_R}\psi(t)\,(t-a)^{p+m}\,dt\\&=\int_{C_{R'}}\psi(t)\,(t-a)^{p+m}\,dt=\int_{C_{R'}}\varphi(t)\,(t-a)^{m-1}\,dt.\end{aligned}$$

Thus the lemma has been proved.

This lemma enables us to consider the integral on a contour on which the integrand has a certain type of singularities.

Let $\varphi(t)$ be the function single valued and analytic on the circle C_R: $|t|=R>0$ and m be a complex number not equal to a positive integer. We define $Y_m(\varphi\,;a)$ by

$$Y_m(\varphi\,;\,a)=\frac{\Gamma(1-m)}{2\pi i}\ \text{Pf.}\int_{C_R}\varphi(t)\,(t-a)^{m-1}\,dt\ :\ a=Re^{i\alpha}, \tag{4}$$

where $(t-a)^{m-1}$ take the principal value if m is not zero or negative integer, and Pf. can be omitted if the real part of m is positive. In the case when m is a positive integer, we define $Y_m(\varphi;\,a)$ by

$$Y_m(\varphi;\,a)=\frac{1}{2\pi i}\int_{C_R}\varphi(t)\,L_m(t-a)\,dt\ ;\ a=Re^{i\alpha}\qquad m=1,2,\ldots, \tag{5}$$

where

$$\begin{cases}L_1(t)=\operatorname{Log} t\\ L_2(t)=t\,(\operatorname{Log} t-1)\\ L_3(t)=\dfrac{t^2}{2!}\left(\operatorname{Log} t-1-\dfrac{1}{2}\right)\\ L_k(t)=\dfrac{t^{k-1}}{(k-1)!}\left(\operatorname{Log} t-1-\dfrac{1}{2}\cdots-\dfrac{1}{k-1}\right),\end{cases} \tag{6}$$

and Log t is the principal value of log t.

The relation

$$Y_m(\varphi';\,a)=Y_{m-1}(\varphi;\,a)=\frac{d}{da}Y_m(\varphi;\,a) \tag{7}$$

is clear from the definition of Y_m by partial integrations. And the operation Y_m is linear in the sense that, for any two functions φ_1 and φ_2 analytic and single valued on C_R,

$$Y_m(\varphi_1+\varphi_2)=Y_m(\varphi_1)+Y_m(\varphi_2)$$

and for φ_n; $n=1, 2, \dots$ analytic on C_R, if φ_n converges to zero uniformly on C_R as n tends to infinity,

$$lim_{n\to\infty} Y_m(\varphi_n)=0.$$

These relations can be verified from the definitions of Y_m and φ_n.

For z interior to C_R, if we put $\varphi(t)=\frac{1}{t-z}$, we have for any complex number m

$$(8)\qquad \begin{cases} Y_m\left(\frac{1}{t-z};\ a\right)=\Gamma(1-m)\,(z-a)^{m-1}\ ;\ m\neq 1, 2, \dots, \\ Y_m\left(\frac{1}{t-z};\ a\right)=L_m(z-a)\qquad ;\ m=1, 2, \dots, \end{cases}$$

where L_m are defined by (6).

Hereafter we denote $Y_m\left(\frac{1}{t-z};\ a\right)$ by $y_m(z\,;\,a)$ for simplicity.

Let $\varphi(t)$ be a function single valued and analytic on and within the circle C_R. Then $Y_m\left(\frac{\varphi(t)}{t-z};\ a\right)$ represents the function $\varphi(z)\,y_m(z;a)$ which is analytic and single valued within C_R but not analytic on C_R, that is, which has a pole or branch point at $z=a$.

2. In this paragraph, we consider the divergence properties of the power series of a function which is analytic interior to the circle C_R and which has singularities of Y_m type on C_R.

At first we consider the following.

Lemma 2. *Let $\varphi(t)$ be a function single valued and analytic on the circle C_R; $|t|=R>0$. Then*

$$(9)\qquad lim_{n\to\infty} n^m a^n Y_m(t^{-n}\varphi(t)\,;\,a)=(-1)^{m-1}a^m\varphi(a).$$

If m is not a positive integer, we have

$$Y_m(t^{-n};\,a)=\frac{\Gamma(1-m)}{2\pi i}\ \mathrm{Pf.}\int_{C_R} t^{-n}(t-a)^{m-1}dt$$

$$=\frac{\Gamma(1-m)}{2\pi i}\int_{C_{R'}\,(R'<R)} t^{-n}(t-a)^{m-1}dt=\frac{\Gamma(1-m)}{(n-1)!}\left[\frac{d^{n-1}}{dt^{n-1}}(t-a)^{m-1}\right]_{t=0}$$

$$=\frac{\Gamma(1-m)}{(n-1)!}(m-1)\,(m-2)\cdots(m-n+1)\,(-a)^{m-n}$$

$$=\Gamma(1-m)\frac{(-m+1)(-m+2)\cdots(-m+n+1)}{1\cdot 2\cdots(n-1)}(-1)^{m-1}(-a)^{m-n}$$
$$\sim(-1)^{m-1}n^{-m}a^{m-n},$$

by the well-known formula

$$lim_{n\to\infty}\frac{1\cdot 2\cdots(n-1)}{z(z+1)\cdots(z+n-1)}=\Gamma(z),$$

where $\sim$ signifies that the ratio of both sides tends to 1 as $n\to\infty$.

If m is a positive integer, we have, for n greater than m,

$$\begin{aligned}Y_m(t^{-n};a)&=\frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dt^{n-1}}L_m(t-a)\right]_{t=0}\\&=\frac{1}{(n-1)!}\left[\frac{d^{n-m-1}}{dt^{n-m-1}}(t-a)^{-1}\right]_{t=0}\\&=\frac{(-1)^{n-m-1}(n-m-1)!}{(n-1)!}(-a)^{m-n}\\&\sim(-1)^{m-1}n^{-m}a^{m-n}.\end{aligned}$$

Thus the validity of (9) can be verified when $\varphi(t)\equiv 1$, that is for any complex number, we have

$$(10)\qquad lim_{n\to\infty}n^m a^n Y_m(t^{-n};a)=(-1)^{m-1}a^m.$$

We are now in a position to prove the lemma for $\varphi(t)$ in general. At first we consider the case when the real part of m is negative. In this case, $\varphi(t)$ can be expanded to

$$\varphi(t)=\sum_{k=0}^{p}\frac{\varphi^{(k)}(a)}{k!}(t-a)^k+(t-a)^{p+1}\psi(t)$$

where $\psi(t)$ is a function single valued and analytic on C_R and p is the largest positive integer such that the real part of $m+p$ is zero or negative. And we have

$$\begin{aligned}n^m a^n Y_m(t^{-n}\varphi(t);a)&=n^m a^n\varphi(a)Y_m(t^{-n};a)\\&+\sum_{k=1}^{p}\frac{\varphi^{(k)}(a)}{k!}\frac{\Gamma(1-m)}{\Gamma(1-m-k)}n^m a^n Y_{m+k}(t^{-n};a)\\&+n^m a^n\frac{\Gamma(1-m)}{\Gamma(-m-p)}Y_{m+p+1}(t^{-n}\psi(t);a),\end{aligned}$$

and if $m+p$ is zero, the last term must be replaced by

$$n^m a^n\frac{\Gamma(1-m)}{2\pi i}\int_{C_R}t^{-n}\psi(t)dt.$$

The first term of the right side members tends to $(-1)^{m-1}a^m\varphi(a)$ and others tend to zeros as n tends to infinity, by (10) and the boundedness of $a^n Y_{m+p+1}(t^{-n}\psi(t);a)$. The relation (9) has been proved in this case.

In the case when the real part of m is zero or positive, putting

$$\varphi(t)=\varphi(a)+\varphi'(a)(t-a)+(t-a)^2\psi(t),$$

we have

$$n^m a^n Y_m(t^{-n}\varphi(t);\, a)=n^m a^n \varphi(a) Y_m(t^{-n};\, a)$$
$$+n^m a^n \varphi'(a) Y_m(t^{-n}(t-a);\, a)+n^m a^n Y_m(t^{-n}(t-a)^2\psi(t);\, a).$$

The first term of the right side members tends to $(-1)^{m-1}a^m\varphi(a)$ and the second tends to zero as $n\to\infty$ by (10). And we can verify that the last term tends to zero as $n\to\infty$ by partial integrations as the $(p+1)$th derivative of $\psi(t)\,(t-a)^{m+1}$ or $\psi(t)\,(t-a)^2 L_m(t-a)$ is continuous, where p is the largest positive integer not greater than the real part of m.

Now the lemma has been established.

Let $\varphi(z)$ be a function single valued and analytic on and within the circle C_R. Partial sums of the power series of the function $\varphi(z)y_m(z;\, a)$ is represented by

$$P_n(z,\, \varphi\, y_m)=Y_m\Big(\frac{t^{n+1}-z^{n+1}}{t^{n+1}}\frac{\varphi(t)}{t-z};\, a\Big); \tag{11}$$
$$n=0,\, 1,\, 2,\, \ldots\, .$$

$P_n(z;\, \varphi y_m)$ are polynomials of respective degrees n, and (11) is valid even for z exterior to C_R.

Theorem 1. *Let $\varphi(z)$ be a function which is single valued and analytic on and within the circle C_R: $|z|=R>0$ and which does not vanish at $z=a$. Let $P_n(z;\, \varphi y_m)$ be partial sums of the power series of $\varphi(z)y_m(z;\, a)$. Then*

$$lim_{n\to\infty} n^m\Big(\frac{a}{z}\Big)^n P_n\,(z;\, \varphi y_m)=A\neq 0 \tag{12}$$

for z exterior to C_R, where A is a complex number non-vanishing and dependent on a, z and φ. Accordingly, $P_n(z;\, \varphi y_m)$ diverges at every point exterior to C_R.

The linearity of Y_m enables us the following calculations. That is, for a point z exterior to C_R,

$$lim_{n\to\infty} n^m\Big(\frac{a}{z}\Big)^n P_n(z;\, \varphi y_m)$$
$$=lim_{n\to\infty}\Big\{ Y_m\Big[n^m\Big(\frac{a}{z}\Big)^n\frac{\varphi(t)}{t-z};\, a\Big]-n^m a^n Y_m\Big[t^{-(n+1)}\frac{\varphi(t)}{t-z};\, a\Big]\Big\}$$
$$=lim_{n\to\infty} -n^m a^n Y_m\Big(t^{-(n+1)}\frac{\varphi(t)}{t-z};\, a\Big).$$

The function $\dfrac{\varphi(t)}{t-z}$ being single valued and analytic on and within the circle C_R and not vanishing at $t=a$, we can verify the relation (12) by lemma 2. Thus the theorem has been established.