155. Dirichlet Problem on Riemann Surfaces. I
(Correspondence of Boundaries)

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Let $\underline{R}$ be an open abstract Riemann surface and let $\{\underline{R}_n\}$ $(n=1, 2, \dots)$ be an exhaustion with compact relative boundaries $\{\partial\underline{R}_n\}$.[1] Then $\underline{R}-\underline{R}_n$ is composed of a finite number of disjoint non compact subsurfaces $\{G_n^i\}$ $(i=1, 2, \dots, i_n$: $n=1, 2, \dots)$. Let $\{G_n^i\}$ be a sequence of non compact surfaces such that $G_n^i \supset G_{n+1}^{i'} \dots$, $\bigcap_n G_n^i=0$. Two sequences $\{G_n^i\}$ and $\{G_m^{i'}\}$ are called equivalent, if and only if, for any given number m, there exists a number n such that $G_m^{i'} \supset G_n^i$ and vice versa. We correspond an ideal point (component) to a class of equivalent sequences and denote the set of all ideal boundary points by B. A topology is introduced on $\underline{R}+B$ by the completion of $\underline{R}$. It is clear that $\underline{R}+B$ is closed, compact and that B is totally disconnected. This topology restricted in $\underline{R}$ is homeomorphic to the original topology. We call this topology A-topology and denote $\underline{R}+B$ by $\underline{R}^*$.[2]

Let R be an abstract Riemann surface given as a covering surface over $\underline{R}$. We define the distance of two points p_1 and p_2 of R by $\inf(\delta(p_1, p_2))$, where $\delta(p_1, p_2)$ is the diameter of the projection of a curve on R connecting p_1 and p_2, and define the accessible boundary points of R by the completion of R with respect to this metric. When a continuous curve L on R converges to the boundary of R and the projection of L on $\underline{R}$ tends to a point of $\underline{R}^*$, we say that L determines an accessible boundary point (abbreviated to A.B.P.). It is well known that these two definitions are equivalent.

In this paper we suppose that $\underline{R}$ is a null-boundary Riemann surface.

Lemma 1.1. Let R be a covering surface over $\underline{R}$, let $\underline{z}=f(z)$ $(\underline{z} \in \underline{R}$, $z \in R)$ be the mapping function from R into $\underline{R}$ and let L be a curve on R which determines an A.B.P. whose projection on B is $\underline{z}_0$. Suppose that R does not cover a subset of positive capacity of $\underline{R}$. We map the universal covering surface R^∞ conformally onto the unit circle U_ξ: $|\xi|<1$ by $\xi=\varphi(z)$. If the image l[3] of L in U_ξ tends to a point ξ_0 on $|\xi|=1$, then the composed function $\underline{z}=f(\varphi^{-1}(\xi))$ has the

1) Thought this paper, we denote a relative boundary of G by ∂G.
2) It is clear that a metric introduced in A-topology.
3) In this case, it is proved that l does not osciliate.

same limit $\underline{z}_0$, *when* ξ *tends to* ξ_0 *along any Stolz's path.*

Proof. Let $\{V_n(\underline{z}_0)\}$ be a sequence of neighbourhoods of $\underline{z}_0$ with compact relative boundaries $\{\partial V_n(\underline{z}_0)\}$ such that $V_n(\underline{z}_0)\supset V_{n+1}(\underline{z}_0)\cdots$ $\bigcap_n V_n(\underline{z}_0)=0$ and let $\underline{R}_0$ be a compact disc of $\underline{R}$ such that the boundary of the projection of R has positive capacity in $\underline{R}_0$. Define a super harmonic function $\omega_n(\underline{z})$ such that $\omega_n(\underline{z})$ is harmonic in $\{(\underline{R}-\underline{R}_0-V_n(\underline{z}_0))\}$ $\cup\{\text{proj}\,(R)\cap\underline{R}_0\}$, $\omega_n(\underline{z}_0)=0$ on the boundary of proj(R) contained in $\underline{R}_0$, $\omega_n(\underline{z})=M_n$ on $\partial\underline{R}_n+V_n(\underline{z}_0)$ and $\frac{1}{2\pi}\int_{\partial\underline{R}n}\frac{\partial\omega_n(\underline{z})}{\partial n}ds=1$. Since $\underline{R}$ is a null-boundary Riemann surface, $\omega_n(\underline{z})$ is uniquely determined and $\lim_n M_n=\infty$. *We denote by* $\Delta_{\xi,\gamma,\delta}$ *the domain:* $|\xi-\xi_0|<1-r$, $|\arg(\xi-\xi_0)|<\frac{\pi}{2}-\delta$ and denote the end part of l outside of $|z|=r$ by l_r. Then we have

$$\omega_n(f(\xi))\geqq M_n\lambda\delta' \text{ on } \Delta_{\xi,\gamma,\delta},\quad n\geqq i_0 \tag{1}$$

where $\lambda\delta'>0$ and i_0 is the minimal number such that $f(l_r)\in V_{i_0}(\underline{z})$. If $f(\xi)$ did not have limit $\underline{z}_0$ in $\Delta_{\xi,\gamma,\delta}$, there would exist a sequence $\{\xi_i\}$ such that $\lim_i \xi_i=\xi_0:\ \xi_i\in\Delta_{\xi,\gamma,\delta}$ and a number n_0 and a sequence $\{\xi_{i'}\}$ such that $f(\xi_{i'})\notin V_{n_0}(z)$; $i'\geqq i_0$. Therefore there exists a number N and i' such that

$$\omega_n(f(\xi_{i''}))\leqq N:\quad i''\geqq i',\ n\geqq n_0. \tag{2}$$

From (1) and (2), we have $N\geqq\omega_m(f(\xi_m))\geqq M_m\lambda\delta'$: $\lim_m M_m=\infty$, which is a contradiction.

If the A.B.P. lies on $\underline{R}$, our assertion is trivial. From this lemma, we can deduce the following.

Lemma 1.2. Under the same conditions as those of the lemma 1.1, let E_ξ *be a set on* $|\xi|=1$. *If the cluster set of* $\underline{z}=f(\xi)$ *on* E_ξ, *when* ξ *tends to points of* E_ξ, *is a set of capacity zero, then* E_ξ *is a set of linear measure zero.*

Proof. From Lemma 1.1, we can suppose that $f(\xi)$ has cluster set in E, along a Stolz's path. We denote by E' a closed set of points p of E' such that $f(\xi)$ tends uniformly when ξ tends to p along a Stolz's path. Then $f(E')\subset E$ is also closed. Let $\{V_n\}$ be a sequence with compact relative boundaries $\{\partial V_n\}$ such that $V_n\supset V_{n+1}$, $\bigcap_n \overline{V}_n=f(E')$, where $\overline{V}$ is the closure of V. Denote by $\omega_n(\underline{z})$ a continuous super-harmonic function such that $\omega_n(\underline{z})$ is harmonic in $(\underline{R}-\underline{R}_0-V_n)\cup(\underline{R}_0\cap\text{proj}\,R)$, $\omega_n(\underline{z})=0$ on $\text{proj}\,R\cup\underline{R}_0$, $\omega_n(\underline{z})=M_n$ on ∂V_n+V_n and $\frac{1}{2\pi}\int\frac{\partial\omega_n(\underline{z})}{\partial n}ds=1$. Then $\lim_n M_n=\infty$. Consider $\omega_n(\underline{z})$ on U_ξ then we have from the super-harmonicity of $\omega_n(\underline{z})$ that $\lim_n\omega_n(\underline{z})\equiv\infty$. This is absurd.

An extension of Fatou's theorem

Lemma 1.3. Let $\underline{R}$ be a null-boundary Riemann surface, let $\underline{O}$, $\underline{\infty}$ be two fixed points of $\underline{R}$ and let a be a point of $\underline{R}$. Denote by $U_a(\underline{z})$ a harmonic function such that

$U_a(\underline{z})+\log \underline{z}=0$ *in the neighbourhood of* $\underline{\infty}$,

$U_a(\underline{z})-\log \underline{z}$ *is harmonic in the neighbourhood of a.*

Then $U_a(\underline{z})$ is determined uniquely. Put $U_a^+(\underline{z})\equiv 0$, if $U_a(\underline{z})\leqq 0$ and $U_a^+(\underline{z})=U_a(\underline{z})$, if $U_a(\underline{z})>0$. Then

$$U_a^+(\underline{z})-d-U_a^+(\underline{O})\leqq U_{\underline{O}}^+(\underline{z}),\quad U_{\underline{O}}^+(\underline{z})\leqq U_a^+(\underline{z})+U_a^+(O)+d,$$

where d depends on $\underline{O}$ and $\underline{\infty}$ only.

Proof. Denote by $D_{\underline{O}}$ the domain such that $0\geqq U_{\underline{O}}(\underline{z}):\ z\in D_{\underline{O}}$ and by $\partial D_{\underline{O}}$ its relative boundary. Put $V(a, \underline{z})=U_a^+(\underline{z})-U_{\underline{O}}^+(\underline{z})-U_a^+(\underline{O})$. Then there are four cases as follows:

1) The case when $a\notin D_{\underline{O}}$, $\underline{z}\notin D_{\underline{O}}$. In this case we have $U_{\underline{O}}^+(\underline{z})=U_{\underline{O}}(\underline{z})$, $U_a^+(\underline{O})=U_a(\underline{O})$ and $V(a, \underline{z})=U_a^+(\underline{z})-U_{\underline{O}}(\underline{z})-U_a(\underline{O})$.
2) The case when $a\in D_{\underline{O}}$, $\underline{z}\notin D_{\underline{O}}$. In this case we have $U_{\underline{O}}^+(\underline{z})=U_{\underline{O}}(\underline{z})$, $U_a^+(\underline{O})=0$ and $V(a, \underline{z})=U_a^+(\underline{z})-U_{\underline{O}}(\underline{z})$.
3) The case when $a\notin D_{\underline{O}}$, $\underline{z}\in D_{\underline{O}}$. In this case we have $U_{\underline{O}}^+(\underline{z})=0$, $U_a^+(\underline{O})=U_a(\underline{O})$ and $V(a, \underline{z})=U_a(\underline{z})-U_a(\underline{O})$.
4) The case when $a\in D_{\underline{O}}$, $z\in D_{\underline{O}}$. In this case we have $U_{\underline{O}}^+(\underline{z})=U_a^+(\underline{O})=0$ and $V(a, \underline{z})=U_a(z)$.

Since $\underline{R}$ has a null-boundary and moreover $V(a, \underline{z})$ is bounded from above and is sub-harmonic with respect to $\underline{z}$ for fixed a and sub-harmonic with respect to a for fixed $\underline{z}$ respectively, $V(a, \underline{z})$ takes its maximum $d=U_{a'}(\underline{z}')=U_{\underline{z}'}(a')$ at some points a' and $\underline{z}'$ on $\partial D_{\underline{O}}$.

In such a case $U_a(\underline{O})=U_{\underline{O}}(a')=U_{\underline{O}}(\underline{z})=0$, hence

$$V(a, \underline{z})\leqq d:\qquad a,\ \underline{z}\in \underline{R}.$$

Therefore, $U_a^+(\underline{z})-d-U_a^+(\underline{O})\leqq U_{\underline{O}}^+(\underline{z})$.

We can prove the latter part similarly.

Let R be a covering surface of positive boundary Riemann surface over $\underline{R}$ and let $\underline{z}=f(z)$: $\underline{z}\in \underline{R}$, $z\in R$ be the mapping function. We denote by $G_n(z, p)$ the Green's function of R_n with pole at p and let $h_n(z, p)$ be its conjugate. Let a_i, b_i the points of R where $f(a_i)=\underline{O}$, $f(b_i)=\underline{\infty}$ respectively. Then we have

$$U_{\underline{O}}(f(z_n))=\frac{1}{2\pi}\int_0^{2\pi}U_{\underline{O}}(f(r_ne^{i\theta_n})d\theta_n+\sum_\nu G_n(z, b_\nu)$$
$$-\sum_\nu G_n(z, a_\nu)+\log |c_k|,$$

where $z_n=e^{-G_n-ih_n}=r_ne^{i\theta_n}$, $G_n(z, O)$ is the Green's function of R with pole at O, and c_k is the first non vanishing coefficient of the expansion of $f(z_n)$ with respect to the local parameter defined in the neighbourhood of O.

Put

$$m(r_n, f-a)=\int_0^{2\pi} U_a^+(f(r_n e^{i\theta_n})d\theta_n,$$

$$N(r_n, f-a)=\int_0^{r_n} \frac{n(t, a)-n(0, a)}{t}dt.$$

We have, by lemma 1.3.

$$|m(r_n, f-a)-m(r_n, \underline{\infty})|\leqq U_Q^+(a)+d,$$

$$m(r_n, a)+N(r_n, a)=m(r_n, \underline{\infty})+N(r_n, \underline{\infty})+\varphi(r_n)$$

where

$$\varphi(r_n)\leqq U_Q(a)+d+|\log|c_k||.$$

If $T(r, \underline{\infty})=\lim_n (N(r_n, \underline{\infty})+m(r_n, \underline{\infty}))<\infty$, we say that $\underline{z}=f(z)$ is a covering of bounded type.

Theorem 1.1. We map the universal covering surface R^∞ of R onto U_ξ : $|\xi|<1$ conformally by $\xi=\varphi(z)$ such that $O=\varphi(O)$. If $\underline{z}=f(z)$ is a covering of bounded type, then the composed function $\underline{z}=\underline{z}(\xi)$ from U_ξ to $\underline{R}$ has angular limits almost everywhere on $|\xi|=1$ and there exists a set E of measure 2π on $|\xi|=1$ such that every Stolz's path terminating at every point of E determines an A.B.P. of R.

Proof. If μ is the equilibrium distribution of unit-mass on a set H of positive capacity on $\underline{R}$, we have $T(r)=\int_H N(r, \underline{z})d\mu(\underline{z})+0(1)$, and $T(r)$ is finite if and only if $N(r, \underline{z})$ is finite everywhere. As to the mapping from R^∞ to U_ξ, we can say that the universal covering surface R_n^∞ of R_n is mapped onto a simply connected domain ${}_\xi D_n$, containing $\xi=0$, and situated in $|\xi|<1$, and that a point a_i of R corresponds to a system of equivalent points $\{a_{ij}\}$ $(j=1, 2, \dots)$: $a_{ij} \in U_\xi$. Let $G_{\xi Dn}(\xi, a_{ij})$ and $G_r(\xi, a_{ij})$ be the Green's function of ${}_\xi D_n$ and $|\xi|<r$ respectively. Then we have

$$G_n(z, a_i)=\sum_j G_{\xi D_n}(\xi, a_{ij}).$$

Since the Green's function is an increasing function of the domain, and since there exists a number n such that ${}_\xi D_n$ contains $|\xi|<r$ for given r, we have

$$G(z, a_i)=\lim_n G_n(z, a_i)\geqq \sum_j G_r(\xi, a_{ij}).$$

This implies that the composed mapping $\underline{z}=f(\varphi^{-1}(\xi))$ is also of bounded type. Put $W(\underline{z})=\exp(U_Q(\underline{z})+iV_Q(\underline{z}))$, where $V_Q(\underline{z})$ is the conjugate function of $U_Q(z)$. A small circle $|W|<\delta$ corresponds to a disc D of $\underline{R}$, whence we have $N(r, a_W)<\infty$ for a_W in D. Hence the analytic function $\exp(U_Q(f(\varphi^{-1}(\xi)))+iV_Q(f(\varphi^{-1}(\xi)))=W(\xi)$ is a function of bounded type in U_ξ. Accordingly $W(\xi)$ has angular limits almost everywhere on $|\xi|=1$ by Fatou's theorem. Since the Green's function $G(z, O)$ of R has angular limit zero almost everywhere on $|\xi|=1$, there exists a set E on $|\xi|=1$ such that $W(\xi)$ has angular limits and

$G(z, O)$ has angular limits zero. Let l be a Stolz's path terminating at $\xi_0 \in E$ such that $W(\xi)$ has limit W_0 along l when ξ tends to ξ_0. We denote by L and $\underline{L}$ the images of l on R and on $\underline{R}$ respectively. We shall prove that L determines an A.B.P.

We see at once that L tends to the boundary of R, because $G(z, O)$ tends to zero along L. Assume that $\underline{L}$ does not converge to a point of $\underline{R}^*$, then we can find two points $\underline{p}_1$ and $\underline{p}_2 \in \underline{R}^*$ and two sequences of points $\{\underline{q}^i{}_1\}$ and $\{\underline{q}^i{}_2\}$ on $\underline{L}$ such that $\lim_i \underline{q}^i = \underline{p}_1$ and $\lim_i q^i{}_2 = \underline{p}_2$. Accordingly we can find a point $p_0 \in \underline{R}$ and a neighbourhood $V(\underline{p}_0)$ such that there exists a sequence of points $\underline{q}_0^{i'}$ on subarc $\lambda_{i'}$ of $\underline{L}$ which tends to $\underline{p}_0$, where $\lambda_{i'}$ is the part of $\underline{L}$ contained in $V(p_0)$ and situated between $q_1^{i'}$ and $\underline{q}_2^{i'}$. Let $W^{i'}(\underline{z})$: $\underline{z} = f(\varphi^{-1}(\xi))$ be the branch of $W(\xi)$ corresponding to $\lambda_{i'}$. Since $|W^{i'}(\underline{z})| = \exp U_{\underline{O}}(\underline{z})$, without loss of generality we can suppose that $\{W^{i'}(\underline{z})\}$ is bounded in $V(\underline{p}_0)$. Thus $\{W^{i'}(\underline{z})\}$ is a normal family. Choose a subsequence $\{W^{i''}(\underline{z})\}$ which converges. Let $W^\infty(\underline{z})$ be its limit function. Since $W^{i''}(\underline{z})$ converges to W_0 on $\lambda_{i''}$ when $i'' \to \infty$, $W^\infty(\underline{z}) \equiv W_0$. On the other hand $|W^i(\underline{z})| = |W(\underline{z})|$ for every i'' and $W(\underline{z}) \not\equiv$ const., which is a contradiction.

Corollary.[4] *Let R be a covering surface over $\underline{R}$. If there exists a set E of positive capacity on $\underline{R}$ such that every point of E is covered by R a finite number of times, then $\underline{z} = f(\varphi^{-1}(\xi))$ has angular limits almost everywhere $|\xi| = 1$.*

In fact, we can easily deduce that in this case $T(r, \underline{\infty})$ is finite.

Corollary. Let $\hat{R}$ be a covering surface over R. If R is a covering of bounded type over $\underline{R}$, then $\hat{R}$ is also of bounded type over $\underline{R}$.

Proof. There exists a system of points $\{a_{ij}\}$ $(j=1, 2, \dots)$ of $\hat{R}$ which lie on a point a_i of R. Since $\sum_j G_{\hat{R}}(\hat{z}, a_{ij}) \leqq G(z, a_i)$, we have at once

$$T(\hat{R}, r) \leqq T(r, \underline{\infty}),$$

where $G_{\hat{R}}(\hat{z}, a_{ij})$ and $G(z, a_i)$ are Green's functions of $\hat{R}$ and R respectively.

4) This corollary has been proved by M. Ohtsuka. See "On a covering surface over an abstract Riemann surface", Nagoya Math. Jour., **4**, 109–118 (1952).