172. The Divergence of Interpolations. II

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Next we shall consider the function analytic interior to the circle C_R and with singularities of Y_m type on C_R. Such functions can be constructed by

$$f(z)=\varphi(z)+\sum_{k=1}^{N}\varphi_k(z)y_{m_k}(z;a_k);\ a_k=Re^{i\alpha_k}, \tag{13}$$

where $\varphi(z)$ and $\varphi_k(z)$ are functions single valued and analytic on and within the circle C_R, and a_k are points on C_R not necessarily distinct. For such functions, we have the following theorem.

Theorem 2. *Let $P_n(z;f)$ be partial sums of the power series of $f(z)$ represented by (13). Then*

$$\overline{lim}_{n\to\infty}\left|n^p\left(\frac{R}{z}\right)^n P_n(z;f)\right|>0 \quad \text{for} \quad |z|>R, \tag{14}$$

where p is the minimal real part of m_k in (13). Accordingly, $P_n(z;f)$ diverges at every point exterior to the circle C_R as n tends to infinity.

In the proof of this theorem, it is convenient to have the following lemma.

Lemma 3. *Let A_k; $k=1, 2, \ldots, N$ be a given set of complex numbers not all equal to zeros. Let α_k; $k=1, 2, \ldots, N$ be mutually distinct angles between zero and 2π, and q_k; $k=1, 2, \ldots, N$ be a set of real numbers. Then we have*

$$\overline{lim}_{n\to\infty}\left|\sum_{k=1}^{N}A_k e^{-i(q_k\log n+n\alpha_k)}\right|>0. \tag{15}$$

For a real number q not equal to zero, the relation

$$e^{-i(q\log n+n\alpha)}=\frac{1}{\Gamma(iq)}\int_0^\infty e^{-n(t+i\alpha)}t^{iq-1}dt;\ n=1, 2, \ldots$$

can be verified by the well-known formula

$$\Gamma(iq)=\int_0^\infty e^{-x}x^{iq-1}dx.$$

Then we have for α not equal to zero

$$\frac{1}{n}\sum_{\nu=1}^{n}e^{-i(q\log\nu+\nu\alpha)}=\frac{1}{n\Gamma(iq)}\int_0^\infty\frac{1-e^{-n(t+i\alpha)}}{1-e^{-(t+i\alpha)}}e^{-(t+i\alpha)}t^{iq-1}dt$$

$$=\frac{1}{\Gamma(iq+1)}\left\{\frac{1}{n}\int_0^\infty\frac{e^{-2(t+i\alpha)}(1-e^{-n(t+i\alpha)})}{[1-e^{-(t+i\alpha)}]^2}t^{iq}dt\right.$$

$$\left.-\int_0^\infty\frac{e^{-(n+1)(t-i\alpha)}}{1-e^{-(t+i\alpha)}}t^{iq}dt+\frac{1}{n}\int_0^\infty\frac{e^{-(t+i\alpha)}(1-e^{-n(t+i\alpha)})}{1-e^{-(t+i\alpha)}}t^{iq}dt\right\}$$

by partial integrations. The last side members converge respectively to zeros as $n\to\infty$, as the minimal absolute value of $1-e^{-(t+i\alpha)}$ is positive for α not equal to zero. Thus we have

$$lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}e^{-i(q\log\nu+\nu\alpha)}=0. \tag{16}$$

And we can verify that this equality is valid even for q equal to zero.

Now we shall prove the lemma. If we assume the equation

$$lim_{n\to\infty}\sum_{k=1}^{N}A_k e^{-i(q_k\log n+n\alpha_k)}=0,$$

we have, for A_1 which can be assumed not to be zero,

$$lim_{n\to\infty}[A_1+\sum_{k=2}^{N}A_k e^{-i\{(q_k-q_1)\log n+n(\alpha_k-\alpha_1)\}}]=0,$$

while the arithmetic means

$$\frac{1}{n}\sum_{\nu=0}^{n-1}[A_1+\sum_{k=2}^{N}A_k e^{-i\{(q_k-q_1)\log\nu+\nu(\alpha_k-\alpha_1)\}}]$$

tend to A_1 as $n\to\infty$ by (16). This contradicts the assumption.

Thus the lemma has been proved.

Now we shall prove Theorem 2. Let p be the minimum of the real part of m_k. From Theorem 1, we have, for z exterior to C_R,

$$\begin{aligned} n^p\left(\frac{R}{z}\right)^n P_n(z;f) &= n^p\left(\frac{R}{z}\right)^n\Big[\frac{1}{2\pi i}\int_{C_R}\frac{t^{n+1}-z^{n+1}}{t^{n+1}}\frac{\varphi(t)}{t-z}dt \\ &\qquad+\sum_{k=1}^{N}Y_{m_k}\Big(\frac{t^{n+1}-z^{n+1}}{t^{n+1}}\frac{\varphi_k(t)}{t-z};\,a\Big)\Big] \\ &\sim\sum_{k=1}^{N}A_k n^{p-m_k}\left(\frac{R}{a_k}\right)^n \\ &=\sum_{k=1}^{N}A_k e^{-i(q_k\log n+n\alpha_k)}n^{p-p_k} \\ &\sim\sum_{k=1}^{N}A_k' e^{-i(q_k\log n+n\alpha_k)}, \end{aligned}$$

where p_k and q_k are respectively the real and imaginal part of m_k, and A_k' are equal respectively to A_k' if $p=p_k$ and $A_k'=0$ if $p<p_k$.

Now the relation (14) follows at once from Lemma 3. The theorem has been established.

3. In this paragraph, we consider the divergence of polynomials which interpolate to such a function considered in the previous paragraph.

Let $f(z)$ be a function single valued and analytic whithin the circle C_R; $|z|=R>1$ and with singularities of Y_m type on C_R. That is, $f(z)$ is represented by (13). Let be given a set of points

$$(17)\qquad \begin{cases} z_1^{(0)} \\ z_1^{(1)},\ z_2^{(1)} \\ z_1^{(2)},\ z_2^{(2)},\ z_3^{(2)} \\ \cdots\cdots\cdots \\ z_1^{(n)},\ z_2^{(n)},\ z_3^{(n)},\ \ldots\ z_{n+1}^{(n)} \\ \cdots\cdots\cdots \end{cases}$$

which do not lie exterior to the unit circle C; $|z|=1$. The sequence of polynomials $S_n(z;f)$ of respective degrees n found by interpolation to $f(z)$ in the points $z_1^{(n)}, z_2^{(n)}, \ldots, z_{n+1}^{(n)}$ is defined by

$$(18)\qquad \begin{cases} S_n(z;f)=\dfrac{1}{2\pi i}\displaystyle\int_{C_R}\frac{W_{n+1}(t)-W_{n+1}(z)}{W_{n+1}(t)}\frac{\varphi(t)}{t-z}dt \\ \qquad\qquad +\sum_{k=1}^{N}Y_{m_k}\left(\dfrac{W_{n+1}(t)-W_{n+1}(z)}{W_{n+1}(t)}\dfrac{\varphi_k(t)}{t-z}dt;\ a_k\right), \end{cases}$$

where

$$W_{n+1}(z)=(z-z_1^{(n)})(z-z_2^{(n)})\cdots(z-z_{n+1}^{(n)}).$$

Let the points (17) satisfy the condition that the sequence $W_n(z)/z^n$ converges to a function non-vanishing and analytic for z exterior to the unit circle. That is, for any real number r greater than unity,

$$(19)\qquad lim_{n\to\infty}\frac{W_n(z)}{z^n}=\lambda(z)\neq 0 \quad \text{uniformly for} \quad |z|\geqq r>1.$$

The sequence of polynomials $S_n(z;f)$ which interpolate to $f(z)$ in all the zeros of $W_{n+1}(z)$ which satisfy the condition (19) has properties similar to those of the power series considered in paragraph 2. At first, we consider the following

Theorem 3. *Let $W_n(z)$ be the sequence of polynomials of respective degrees n such that the sequence $W_n(z)/z^n$ converges to a function $\lambda(z)$ non-vanishing and analytic for z exterior to the unit circle C; $|z|=1$, and uniformly on any finite closed set exterior to C. Let $\varphi(z)$ be a function single valued and analytic on and within the circle C_R. Then the sequence of polynomials $S_n(z;\varphi y_m)$ of respective degrees n which interpolate to $\varphi(z)y_m(z;a)$ in all the zeros $W_{n+1}(z)$ converges to $\varphi(z)y_m(z;a)$ for z interior to C_R. The sequence $S_n(z;\varphi y_m)$ diverges at every point exterior to C_R. Moreover, we have*

$$(20)\qquad lim_{n\to\infty}n^m\left(\frac{a}{z}\right)^n S_n(z;\varphi y_m)=B\neq 0.$$

The first part of the theorem, that is the convergence of $S_n(z;\varphi y_m)$ has been well known. (Cf. Walsh: Interpolations and Approximations, American Mathematical Society Colloquium Publications (1935).)

Now we shall prove the relation (20). By the method similar to the proof of Theorem 1, we have

$$n^m\left(\frac{a}{z}\right)^{n+1} S_n(z;\, \varphi y_m)=n^m\left(\frac{a}{z}\right)^{n+1} Y_m\left(\frac{W_{n+1}(t)-W_{n+1}(z)}{W_{n+1}(t)}\frac{\varphi(t)}{t-z};\, a\right)$$

$$\sim -n^m a^{n+1}\frac{W_{n+1}(z)}{z^{n+1}}\, Y_m\left(\frac{1}{W_{n+1}(t)}\frac{\varphi(t)}{t-z};\, a\right)$$

$$\sim -n^m a^{n+1}\lambda(z)\, Y_m\left(t^{-(n+1)}\frac{\varphi(t)}{\lambda(t)(t-z)};\, a\right)$$

$$+n^m a^{n+1}\lambda(z)\, Y_m\left[t^{-(n+1)}\left(\frac{1}{\lambda(t)}-\frac{t^{n+1}}{W_{n+1}(t)}\right)\frac{\varphi(t)}{t-z};\, a\right].$$

We can verify that the last term of the last members tends to zero as $n\to\infty$ by the method used in the proof of Lemma 2, that is by partial integrations as

$$\frac{d^j}{dt^j}\left\{\frac{1}{\lambda(t)}-\frac{t^{n+1}}{W_{n+1}(t)}\right\};\quad j=1, 2, \ldots, p$$

converge respectively to zeros uniformly on C_R as n tends to infinity. And the first term of the last side members tends to

$$\lambda(z)a^m\frac{\varphi(a)}{\lambda(a)(a-z)}=B\neq 0 \quad \text{for} \quad |z|>R.$$

Thus the theorem has been established.

Next we shall consider the behavior of $S_n(z;f)$ which interpolate to $f(z)$ defined by (13) in all the zeros of $W_{n+1}(z)$.

Theorem 4. *Let $f(z)$ be the function represented by (13). Let $W_n(z)$ be the sequence of polynomials of respective degrees n such as the sequence $W_n(z)/z^n$ converges to a function $\lambda(z)$ non-vanishing and analytic for z exterior to the unit circle C, and uniformly on any finite closed set exterior to C. Then the sequence of polynomials $S_n(z;f)$ of respective degrees n which interpolate to $f(z)$ in all the zeros of $W_{n+1}(z)$ converges to $f(z)$ for z interior to C_R. The sequence $S_n(z;f)$ diverges at every point exterior to C_R. Moreover, we have*

$$\overline{\lim}_{n\to\infty}\left|n^p\left(\frac{R}{z}\right)^n S_n(z;f)\right|>0;\quad |z|>R>1, \tag{21}$$

where p is the minimal real part of m_k in (13).

The validity of the relation (21) is sufficient to prove the theorem. Let p be the minimum of the real part of m_k in (13). We have, from (18) and Theorem 3 by the method similar to the proof of Theorem 2,

$$n^p\left(\frac{R}{z}\right)^{n+1}S_n(z;f)=n^p\left(\frac{R}{z}\right)^{n+1}\frac{1}{2\pi i}\int_{C_R}\frac{W_{n+1}(t)-W_{n+1}(z)}{W_{n+1}(t)}\frac{\varphi(t)}{t-z}dt$$

$$+\,n^p\left(\frac{R}{z}\right)^{n+1}\sum_{k=1}^{N}Y_{m_k}\left(\frac{W_{n+1}(t)-W_{n+1}(z)}{W_{n+1}(t)}\frac{\varphi_k(t)}{t-z};\,a_k\right)$$

$$\sim -n^pR^{n+1}\lambda(z)\sum_{k=1}^{N}\left(t^{-(n+1)}\frac{\varphi_k(t)}{\lambda(t)(t-z)};\,a_k\right)$$

$$\sim\sum_{k=1}^{N}B_kn^{p-m_k}\left(\frac{R}{a_k}\right)^n$$

$$=\sum_{k=1}^{N}B_ke^{-i(q_k\log n+n\alpha_k)}n^{p-p_k}$$

$$\sim\sum_{k=1}^{N}B_k'e^{-i(q_k\log n+n\alpha_k)},$$

where p_k and q_k are respectively the real and imaginal part of m_k and B_k' are equal respectively to B_k if $p=p_k$, and $B_k'=0$ if $p<p_k$.

Now the relation (21) follows at once by Lemma 2. Thus the theorem has been established.