201. *Harmonic Measures and Capacity of Sets of the Ideal Boundary. I*

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(Comm. by K. KUNUGI, M.J.A., Dec. 13, 1954)

Let R be an abstract Riemann surface of positive boundary and let $\{R_n\}$ $(n=0,1,2,\dots)$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$.[1] Each ∂R_n consists of a finite number of analytic curves. Let D be a non compact subdomain whose relative boundary ∂D consists of at most an enumerably infinite number of analytic curves clustering nowhere in R. We say that a sequence $\{D\cap(R-R_n)\}$ determines a subset of the ideal boundary, which is denoted by B_D. In this article we shall introduce the harmonic measures and capacity of B_D and study their applications.

1. Harmonic Measures

Let $U(z)$ be a continuous function in R. If there exists a number n such that $U(z)\geqq 1-\varepsilon$ for given ε in $D\cap(R-R_n)$, we say that $U(z)$ has limit $\geqq 1$ in B_D. Let $\omega_{n,n+i}(z)$ be a bounded harmonic function in $R_{n+i}-((R_{n+i}-R_n)\cap D)$ such that $\omega_{n,n+i}(z)=0$ on $\partial R_{n+i}-D$ and $\omega_{n,n+i}(z)=1$ on $(\partial R_n\cap D)+(\partial D\cap R_{n+i})$. Then $\omega_{n,n+i+j}(z)\geqq\omega_{n,n+i}(z)$ and $\omega_{n+i,j}(z)\leqq\omega_{n,j}(z)$. Put $\lim_{n=\infty}\lim_{i=\infty}\omega_{n,n+i}(z)=\omega(z)$. We call $\omega(z)$ the outer harmonic measure of B_D. We define the inner harmonic measure of B_D similarly. Another definition is as follows: Let $\{v(z)\}$ be a class of continuous super-harmonic functions such that $0\leqq v(z)\leqq 1$, $\lim v(z)\geqq 1$ in B_D. Let $V(z)$ be its lower envelope. Then it is easy to prove that $V(z)=\omega(z)$. Let R_0 be a compact disc in R and let $\omega'_{n,n+i}(z)$ be a bounded harmonic function in $R_{n+i}-((R_{n+i}-R_n)\cap D)-R_0$ such that $\omega'_{n,n+i}(z)=0$ on $\partial R_0+(\partial R_{n+i}-D)$ and $\omega'_{n,n+i}(z)=1$ on $(\partial R_n\cap D)+(\partial D\cap R_{n+i})$.

Then $\lim_{n=\infty}\lim_{i=\infty}\omega'_{n,n+i}(z)=\omega'(z)$. We have at once from the definition the follwing

Theorem 1. *Let B_{D_1} and B_{D_2} be two subsets of ideal boundary and let $\omega_{D_i}(z)$ be harmonic measures of B_{D_i}. Then*

$$\omega_{D_1}(z)+\omega_{D_2}(z)\geqq\omega_{D_1+D_2}(z),\quad \omega'_{D_1}(z)+\omega'_{D_2}(z)\geqq\omega'_{D_1+D_2}(z).$$

If $D'\supset((R-R_m)\cap D)$ for a number m, we say that D' covers B_D. Let $D_1\supset D_2,\dots$ be a sequence of non compact domains containing B_D and let $U(z)$ be a positive harmonic function in R. We denote

1) In this article, we denote by ∂G the relative boundary of G.

the lower envelope of continuous super-harmonic functions $\{v(z)\}$ such that $v(z)\geqq U(z)$ in D by $V_D(z)$. Then $V_{D_1}(z)\geqq V_{D_2}(z)\cdots$ and $\lim_{i=\infty} V_{D_i}(z)=V_D(z)=U_{ex}(z)$. We say that $V_D(z)$ is obtained from $U(z)$ by the extremisation with respect to $\{D_i\}$.

Theorem 2

$$(U_{ex}(z))_{ex}=U_{ex}(z).$$

Lemma 1. *Let $V_D(z)$ be the lower envelope of non negative continuous super-harmonic functions $\{v(z)\}$ such that $v(z)\geqq U(z)$ in D and let $V_G(z)$ be the lower envelope of continuous super-harmonic functions $\{v(z)\}$ such that $v(z)\geqq V_D(z)$ in a non compact domain $G: G\supset D$. Then*

$$V_G(z)=V_D(z).$$

Proof. Let $v_D^n(z)$ be a harmonic function in R_n-D such that $v_D^n(z)=U(z)$ on $\partial D\cap R_n$ and $v_D^n(z)=0$ ∂R_n-D. Then $v_D^n(z)\uparrow V_D(z)$. On the other hand let $v_G^n(z)$ be a harmonic function in R_n-G such that $v_G^n(z)=V_D(z)$ on $\partial G\cap R_n$ and $v_G^n(z)=0$ on ∂R_n-G. Then $v_G^n(z)\uparrow V_G(z)$. Since $v_D^n(z)\uparrow V_D(z)$, $v_G^n(z)\geqq v_D^n(z)$ on $\partial G\cap R_n$ and $v_D^n(z)=v_G^n(z)=0$ on ∂R_n-G, whence $V_G(z)\geqq V_D(z)$. On the other hand, it is clear $v_G^n(z)\leqq V_D(z)$. Hence $V_G(z)=V_D(z)$.

Lemma 2. *Let $\varphi_i(z)$ $(i=1,2,\ldots)$ $(\varphi_i\leqq\varphi^*)$ be positive continuous boundary functions on ∂G such that $\int\varphi^*\frac{\partial G(z,p)}{\partial n}ds<\infty$, where $G(z,p)$ is the Green's function of $R-G$. Let $V_{\varphi_i}(z)$ be the lower envelope of non negative continuous super-harmonic functions $\{v(z)\}$ such that $v(z)\geqq\varphi_i(z)$ on ∂G. If $\varphi_i(z)\to\varphi(z)$ on ∂G, then $V_{\varphi_i}(z)\to V_\varphi(z)$.*

Let $G_n(z,p)$ be the Green's function of R_n-G. Then $G_n(z,p)\uparrow G(z,p)$ and $\frac{\partial G_n(z,p)}{\partial n}\uparrow\frac{\partial G(z,p)}{\partial n}$ on ∂G. Hence

$$2\pi=\lim_{n=\infty}\int_{\partial G\cap R_n}\frac{\partial G_n(z,p)}{\partial n}ds=\int_{\partial G}\lim\frac{\partial G_n(z,p)}{\partial n}ds.$$

For any given number $\varepsilon>0$, we can find a number n_0 and k such that $|\varphi(z)-\varphi_k(z)|<\varepsilon$ on $\partial G\cap R_n$ and $\int_{\partial G_n\cap(R-R_n)}\varphi^*\frac{\partial G(z,p)}{\partial n}ds<\varepsilon$. Then $|V_{\varphi_i}^n(z)-V_\varphi^n(z)|<3\varepsilon(n>n_0,k>k_0(n_0))$, where $V_{\varphi_i}^n(z)$ and $V_\varphi^n(z)$ is harmonic functions in R_n-G which have boundary values φ_i and φ on $\partial G-R_n$ and vanish on ∂R_n-G. Let $\varepsilon\to 0$. We have $\lim_{i=\infty}V_{\varphi_i}=V_\varphi(z)$.

Let $\tilde V_m^{m+i}(z)$ and $\tilde V_m(z)$ be the lower envelopes of continuous super-harmonic functions in $R-D_m$ which have as their boundary values on ∂D_m, $V_{Dm+i}(z)$ and $U_{ex}(z)$ respectively. Then by Lemma 1, $\tilde V_m^{m+i}(z)=V_{Dm+i}(z)$. Since $\lim_{i=\infty}V_{D_i}(z)=U_{ex}(z)$, $\lim_{j=\infty}V_{Dm+j}(z)=U_{ex}(z)$ on ∂D_m, then by Lemma 2, we have $(U_{ex}(z))_{ex}=\lim_{m=\infty}\lim_{i=\infty}\tilde V_m^{m+i}(z)=\lim_{i=\infty}V_{Dm+i}(z)=U_{ex}(z)$.

Corollary. *Let $\omega(z)$ be the outer harmonic measure of B_D. Then if $\omega(z)\neq 0$, $\underline{\lim}_{z\in D}\omega(z)=1$, and if $\omega'(z)\neq 0$, $\overline{\lim}_{z\in D}\omega(z)=1$.*

Proof. We can easily prove as in the proof of Theorem 2 that $\omega(z)=\omega_{ex}(z)$. Let $\tilde{\omega}_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-((R-R_n)\cap D)$ such that $\tilde{\omega}_{n,n+i}(z)=\omega(z)$ on $\partial D\cap(R_{n+i}-R_n)$ and $\tilde{\omega}_{n,n+i}(z)=0$ on $\partial R_{n+i}-D$. If $\overline{\lim}_{z\in D}\omega(z)\leqq K<1$, $\tilde{\omega}_{n,n+i}(z)\leqq K\omega_{n,n+i}(z)$, where $\lim_{n=\infty}\lim_{i=\infty}\omega_{n,n+i}(z)=\omega(z)$. But $\lim_{n=\infty}\lim_{i=\infty}\tilde{\omega}_{n,n+i}(z)=\omega_{ex}(z)=\omega(z)$ implies $\omega_{ex}(z)\leqq K\omega_{ex}(z)$. Hence $\omega_{ex}(z)=0$. This is absurd. The latter part is proved similarly.

Corollary

$$\omega'(z)=0 \quad \textit{is equivalent to} \quad \omega(z)=0.$$

Proof. Let $\hat{\omega}_{n,n+i}(z)$ be the harmonic measure of ∂R_0 with respect to $R_{n+i}-(D\cap(R_{n+i}-R_n))-R_0$ i.e. $\hat{\omega}_{n,n+i}(z)=1$ on ∂R_0 and $\hat{\omega}_{n,n+i}(z)=0$ on $(\partial R_{n+i}-D)+(\partial D\cap(R_{n+i}-R_n))+(\partial R_n\cap D)$. Suppose $\omega'(z)=0$. Then $\hat{\omega}_{n,n+i}(z)+\omega'_{n,n+i}(z)\geqq\omega_{n,n+i}(z)$. Let $i\to\infty$ and then $n\to\infty$. Then we have $\hat{\omega}(z)\geqq\omega(z)$, where $\hat{\omega}(z)\neq 1$, because R is a positive boundary Riemann surface. Denote the maximum of $\hat{\omega}(z)$ on ∂R_1 by $\lambda\,(\lambda<1)$. Since $\hat{\omega}(z)<\lambda$ in $R-R_1$, $\overline{\lim}_{z\in D}\omega(z)\leqq\lambda<1$, whence $\omega(z)=0$. On the other hand $\omega(z)>\omega'(z)$, $\omega(z)=0$ implies $\omega'(z)=0$.

2. Capacity

Let $U_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-R_0-(D\cap(R_{n+i}-R_n))=B_{n,n+i}$ such that $U_{n,n+i}(z)=0$ on ∂R_0, $U_{n,n+i}(z)=1$ on $(\partial R_n\cap D)+((\partial D\cap(R_{n+i}-R_n))$ and $\frac{\partial U_{n,n+i}}{\partial n}=0$ on $\partial R_{n+i}-D$. Then we have $D_{B_{n,n+i}}(U_{n,n+i}(z)-U_{n,n+i+j}(z),\,U_{n,n+i}(z))=0$, whence

$$D_{B_{n,n+i}}(U_{n,n+i+j}(z))=D_{B_{n,n+i}}(U_{n,n+i}(z))+D_{B_{n,n+i}}(U_{n,n+i}(z)-U_{n,n+i+j}(z)).$$

But clearly $D_{B_{n,n+i}}(U_{n,n+i}(z))\leqq D_{R-R_0}(U^*(z))$ for every i and n, where $U^*(z)$ is a harmonic function in R_1-R_0 such that $U^*(z)=0$ on ∂R_0 and $U^*(z)=1$ on ∂R_1. From the above consideration

$$\lim_{\substack{i=\infty\\ j=\infty}} D_{B_{n,n+i}}(U_{n,n+i}(z)-U_{n,n+i+j}(z))\leqq\lim_{\substack{i=\infty\\ j=\infty}} D_{B_{n,n+i}}(U_{n,n+i}(z)-U_{n,n+i+j}(z))=0.$$

Thus $\{U_{n,n+i}(z)\}$ converges in mean, since $U_{n,n+i}(z)=0$ on ∂R_0, it converges to $U_n(z)$ uniformly in every compact set of $R-(D\cap(R-R_n))$. We see by the maximum principle that

$$U_{n+i,n+i+j}(z)\leqq U_{n,n+i}(z),\ \text{whence}\ \lim_{j=\infty}U_{n+i,n+i+j}(z)=U_{n+i}(z)\leqq U_n(z)$$

$=\lim_{j=\infty}U_{n,n+i+j}(z)$. Thus $\{U_n(z)\}$ converges to a harmonic function denoted by $U(z)$.

Put $Cap\,(B_D)=\lim_{n=\infty}\int_{\partial R_0}\frac{\partial U_n(z)}{\partial n}ds=\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds.$ We call it the capacity of B_D and $U(z)$ the equilibrium potential.

Lemma 1. *Let G be a non compact domain containing D and let $U_D(z)$ be the harmonic function which has the minimum Dirichlet integral over $R-D-R_0$ among all function $\{U(z)\}$ which have the same boundary value φ on $\partial D+\partial R_0$ and let $U_G(z)$ be the function with minimum Dirichlet integral over $R-G-R_0$ which has the boundary value $U_D(z)$ on $\partial R_0+\partial G$. Then*

$$U_D(z)=U_G(z).$$

Proof. Let $U'_n(z)$ be a harmonic function in R_n-G-R_0 such that $U'_n(z)=U_D(z)$ on $\partial G\cap R_n$ and $\frac{\partial U'_n}{\partial n}=0$ on ∂R_n-G. We see $\{U'_n(z)\}$ converges. Put $\lim\limits_{n=\infty}U'_n(z)=U'(z)$.

Assume $D_{R-G}(U'(z))\leqq D_{R-G}(U_D(z))-d(d>0)$. Then $D_{R_n-G}(U'_n(z))\leqq D_{R-G}(U_D(z))-d-\varepsilon$. Let $U''_n(z)$ be a harmonic function such that $U''_n(z)=U_D(z)$ on $\partial R_n\cap(G-D)$ and $U''_n(z)=U'_n(z)$ on ∂R_n-G. Then $D_{R_n-D}(U''_n(z))\leqq D_{R_n-G}(U'_n(z))+D_{R_n\cap(G-D)}(U_D(z))\leqq D_{R_n-D}(U_D(z))-d$.

Choose a sequence $\{U''_{n'}(z)\}$ of $\{U''_n(z)\}$ which converges to $U^*(z)$. We have $D_{R-D}(U^*(z))\leqq D_{R-D}(U'(z))-d$. This contradicts the minimality of $D_{R-D}(U_D(z))$. Hence $D_{R-G}(U'(z))=D_{R-G}(U_D(z))$. We also see that $U'(z)$ is a harmonic continuation of $U_D(z)$ by the Dirichlet principle. On the other hand, since $D_{R_n-G}(U_D(z)-U'_n(z),\ U'_n(z))=0$, we have $D_{R_n-G}U_D(z)-U'_n(z))\leqq D_{R_n-G}(U_D(z))-D_{R_n-G}(U'_n(z))$, whence $U_G(z)=U_D(z)$.

Let $U_{n,n+i}(z)$ be the harmonic function in $R_{n+i}-R_0-(D\cap(R_{n+i}-R_n))$ defined above. Put $U_n(z)=\lim\limits_{i=\infty}U_{n,n+i}(z)$. We denote the domain where $U_n(z)>1-\varepsilon$ by G_ε.[2] It is clear $G_\varepsilon\supset(D\cap(R-R_n))$. Denote by $U_{\varepsilon,i}(z)$ a bounded harmonic function in $R_{n+i}-R_0-G_\varepsilon$ such that $U_{\varepsilon,i}(z)=0$ on ∂R_0, $U_{\varepsilon,i}(z)=1-\varepsilon$ on $\partial G_\varepsilon\cap(R_{n+i}-R_n)$ and $\frac{\partial U_{\varepsilon,i}}{\partial n}=0$ on $\partial R_{n+i}\cap(R-G_\varepsilon)$. Then $U_{\varepsilon,i}(z)$ converges to $U_n(z)$ and

$$D_{R_{n+i}-G_\varepsilon}(U_{\varepsilon,i}(z))=(1-\varepsilon)\int\limits_{\partial R_0}\frac{\partial U_{\varepsilon,i}}{\partial n}ds=\int\limits_{\partial G_\varepsilon\cap R_{n+i}}\frac{\partial U_{\varepsilon,i}}{\partial n}ds.$$

Let $i\to\infty$. Since $D_{R_{n+i}-G_\varepsilon}(U_{\varepsilon,i}(z))\uparrow D_{R-G_\varepsilon}(U_n(z))$, $D_{R-G_\varepsilon}(U_n(z))=(1-\varepsilon)\int\limits_{\partial R_0}\frac{\partial U_n}{\partial n}ds$. Since $\frac{\partial U_{\varepsilon,i}}{\partial n}\leqq 0$ on ∂G_ε, by Fatou's lemma

$$0\geqq\int\limits_{\partial G_\varepsilon\cap R}\lim_{i=\infty}\frac{\partial U_{\varepsilon,i}}{\partial n}\geqq\lim_{i=\infty}\int\limits_{\partial G_\varepsilon}\frac{\partial U_{\varepsilon,i}}{\partial n}ds,\ \text{thus}\int\limits_{\partial R_n\cap(R-G_\varepsilon)}\frac{\partial U_n}{\partial n}ds\leqq 0.$$

Lemma 2

$$\int\limits_{\partial G_\varepsilon}\frac{\partial U_n}{\partial n}ds=\int\limits_{\partial R_0}\frac{\partial U_n}{\partial n}ds,$$

2) The niveau curve $C_n^\varepsilon=\underset{z}{\mathfrak{E}}\{U_n(z)=1-\varepsilon\}$ may be compact for every ε. We imagine that such case occurs on some Riemann surfaces of O_{HD}.

except for at most one ε.

To prove the lemma, we show $\lim_{m=\infty}\int_{\partial R_m\cap(R-G_\varepsilon)}\frac{\partial U_n}{\partial n}ds=0$, for a subsequence $\{R_{m'}\}$. If there were two constants ε_1 and $\varepsilon_2(\varepsilon_1<\varepsilon_2)$ such that $\lim_{m'=\infty}\int_{\partial R_{m'}\cap(R-G_{\varepsilon_1})}\frac{\partial U_n}{\partial n}ds=P_1<0$ and $\lim_{m'=\infty}\int_{\partial R_{m'}\cap(R-G_{\varepsilon_2})}\frac{\partial U_n}{\partial n}ds=P_2<0$. Consider the Dirichlet integral

$$D_{R-G_{\varepsilon_1}}(U_n(z))=D_{R-G_{\varepsilon_1}}(1-\varepsilon_1-U_n(z))=(1-\varepsilon_1)\int_{\partial R_0}\frac{\partial U_n(z)}{\partial n}ds$$

$$+\left[\lim_{i=\infty}\int_{(R-G_{\varepsilon_2})\cap\partial R_{m+i'}}(1-\varepsilon_1-U_n(z))\frac{\partial U_n}{\partial n}ds+\int_{(G_{\varepsilon_1}-G_{\varepsilon_2})\cap\partial R_{m+i'}}(1-\varepsilon_1-U_n(z))\frac{\partial U_n}{\partial n}ds\right].$$

Since $P_1<0$ and $P_2<0$, $P_2-P_1>P_2+\delta\,(\delta>0)$.

Hence
$$\int_{(R-G_{\varepsilon_2})\cap\partial R_{m+i'}}(1-\varepsilon_1-U_n(z))\frac{\partial U_n}{\partial n}ds\leqq P_2(\varepsilon_2-\varepsilon_1)$$

and
$$\int_{(G_{\varepsilon_2}-G_{\varepsilon_1})\cap\partial R_{m+i'}}(1-\varepsilon_1-U_n(z))\frac{\partial U_n}{\partial n}ds\leqq(-\delta-P_2)(\varepsilon_2-\varepsilon_1).$$

Therefore

$$(1-\varepsilon_1)\int_{\partial R_0}\frac{\partial U_n}{\partial n}ds=D_{R-G_{\varepsilon_1}}(U_n(z))=D_{R-G_{\varepsilon_1}}(U_n(z))+P_2(\varepsilon_2-\varepsilon_1)$$
$$+(-\delta-P_2)(\varepsilon_2-\varepsilon_1)\lneqq D_{R-G_{\varepsilon_1}}(U_n(z)).$$

This is absurd.

Hence $\lim_{i=\infty}\int_{(R-G_\varepsilon)\cap\partial R_{m+i}}\frac{\partial U_n}{\partial n}ds=0$ for ε except at most for one ε, whence

$$\int_{\partial G_\varepsilon}\frac{\partial U_n}{\partial n}ds=\int_{\partial R_0}\frac{\partial U_n}{\partial n}ds=\lim_{i=\infty}\int_{\partial R_{m+i}\cap\partial G_\varepsilon}\frac{\partial U_{\varepsilon,i}}{\partial n}ds. \qquad (1)$$

Let $\omega_{n,n+i}(z)$ be a harmonic function in $R_{m+i}-G_\varepsilon$ such that $0\leqq\omega_{m,m+i}(z)\leqq1$, $\omega_{m,m+i}(z)=0$ on $R_m\cap\partial G_\varepsilon+\partial R_0$, $\omega_{m,m+i}(z)=1$ on $(R_{m+i}-R_n)\cap\partial G_\varepsilon$ and $\frac{\partial\omega_{m,m+i}}{\partial n}=0$ on $\partial R_{n+i}-G_\varepsilon$. Then we can prove that $\omega_{m,m+i}(z)$ converges to $\omega_m(z)$, where ε is the number satisfying (1). Then

$$\int_{\partial R_0+(\partial G_\varepsilon\cap R_{n+i})+(\partial R_{n+i}-G_\varepsilon)}U_{\varepsilon,i}(z)\frac{\partial\omega_{m,m+j}}{\partial n}ds=\int_{\partial R_0+(\partial G_\varepsilon\cap R_{n+i})+(\partial R_{n+i}-G_\varepsilon)}\omega_{m,m+j}(z)\frac{\partial U_{\varepsilon,i}}{\partial n}ds,$$

where $n+i=m+j$ and $m>n$.

Hence
$$\int_{\partial R_0}(1-\varepsilon)\frac{\partial\omega_{m,m+t}}{\partial n}ds=\int_{(R_{n+i}-R_m)\cap\partial G_\varepsilon}\frac{\partial U_{\varepsilon,i}}{\partial n}ds$$

implies
$$(1-\varepsilon)\int_{\partial R_0}\frac{\partial\omega_m}{\partial n}ds=\lim_{i=\infty}\int_{(R_{n+i}-R_m)\cap\partial G_\varepsilon}\frac{\partial U_{\varepsilon,i}}{\partial n}=\int_{(R-R_m)\cap\partial G_\varepsilon}\frac{\partial U_n}{\partial n}ds.$$

Thus $$\lim_{m=\infty} \omega_m(z) = 0.$$

Extremisation

Let D be a non compact domain defining B_D and put $D_n = D \cap (R - R_n)$. Let $U_n(z)$ be the harmonic function in $R - D_n - R_0$ such that $U_n(z)=1$ on ∂D_n, $U_n(z)=0$ on ∂R_0 and $U_n(z)$ has the minimum Dirichlet integral over $R-R_0-D_n$. Then $U_n(z) \geqq U_{n+i}(z)$. Since $U_n(z) \geqq U_{n+i}(z)$, $D_n + G_n \supset D_{n+1} + G_{n+1}, \ldots$; $\varepsilon_1 \geqq \varepsilon_2 \geqq; \ldots$; $\lim_{n=\infty} \varepsilon_n = 0$, where G_n is the domain such that $U_n(z) > 1 - \varepsilon_n$ in G_n and every ε_n satisfies (1). Let $V(z)$ be a bounded harmonic function in $R-R_0$ such that $D_{R-R_0}(V(z)) < \infty$ and $V(z)=0$ on ∂R_0 and let $V_m(z)$ be a harmonic function in $R-R_0-(D_m+G_m)$ such that $V(z)=V_m(z)$ on ∂G and $V_m(z)$ has the minimum Dirichlet integral. It is clear $V_m(z) = \lim_{i=\infty} V_{m,m+i}(z)$, where $V_{m,m+i}(z)$ is a harmonic function in $R_{m+i} - R_0 - G_m$ such that $V_{m,m+i}(z) = V(z)$ on $\partial G_m \cap R_{m+i}$ and $\frac{\partial V_{m,m+i}}{\partial n} = 0$ on $\partial R_{m+i} - G_m$. Since $D_{R-G_m}(V_m(z)) < D_{R-R_0}(V(z))$, we can choose a subsequence $\{V_{m'}(z)\}$ which converges to $V_{ex}^{\gamma}(z)$ in every compact domain in R. We say that $V_{ex}^{\gamma}(z)$ is obtained from $V(z)$ by extremisation with respect to γ sequence $\{G_{m'}\}$.

Theorem 3

$$V_{ex}^{\gamma}(z) = (V_{ex}^{\gamma}(z))_{ex}.$$

Proof. Let $\omega_{m,j,k}(z)$ be a harmonic function such that $\omega_{m,j,k}(z)=0$ on $\partial G_m \cap R_{m+j}$, $\omega_{m,j,k}(z)=1$ on $\partial G_m \cap (R_{m+j+k} - R_{m+j})$ and $\frac{\partial \omega_{m,j,k}}{\partial n} = 0$ on $\partial R_{m+j+k} - G_m$. Put $\lim_{k=\infty} \omega_{m,j,k}(z) = \omega_{m,j}(z)$, then by Lemma 2, there exists j_0 such that $\omega_{m,j}(p) < \frac{\varepsilon}{M} (j \geqq j_0(p))$ for any given p, where $M = \overline{\lim}_{\substack{z \in R \\ m=1,2\ldots}} \{V_m(z)\}$. Since $\{V_{m'}(z)\}$ converges to $V_{ex}^{\gamma}(z)$, for G_m and for any given number ε, there exists $l_0 = l_0(j_0, p)$ such that

$$|V_{l'}(z) - V_{ex}^{\gamma}(z)| < \varepsilon \text{ on } \partial G_m \cap R_{m+j} (l > l_0) \text{ and } \omega_{m,j}(p) < \frac{\varepsilon}{M}.$$

On the other hand $V_l(z) = \lim_{k=\infty} V_{l,k}(z)$, where $V_{l,k}(z)$ is a harmonic function such that $V_{l,k}(z) = V_l(z)$ on $\partial G_m \cap R_{m+k}$, $\frac{\partial V_{l,k}}{\partial n} = 0$ on $\partial R_{m+k} - G_m$. Let $\tilde{V}_{m,j}(z)$ be a harmonic function such that $\tilde{V}_{m,k}(z) = V_{ex}^{\gamma}(z)$ on $\partial G_m \cap R_{m+k}$ and $\frac{\partial \tilde{V}_{m,k}}{\partial n} = 0$ on $\partial R_{m+k} - G_m$. Then $V_{l,k}(z) \leqq \tilde{V}_{m,k}(z) + \varepsilon + 2M\omega_{m,j,k}(z)$, let $k \to \infty$. Then we have $|V_l(p) - \tilde{V}_m(p)| < 3\varepsilon$ and let $l \to \infty$ and $\varepsilon \to 0$. Then $V_{ex}(z) - \tilde{V}_m(z) \leqq 0$. The inverse inequality will be obtained as above. Since $\lim_{m=\infty} \tilde{V}_m(z) = (V_{ex}^{\gamma}(z))_{ex}$, thus $V_{ex}^{\gamma}(z) = (V_{ex}^{\gamma}(z))_{ex}$.