# 200. *Dirichlet Problem on Riemann Surfaces.* IV

## (*Covering Surfaces of Finite Number of Sheets*)

By Zenjiro KURAMOCHI
Mathematical Institute, Osaka University

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Let $\underline{R}$ be a null-boundary Riemann surface with $A$-topology and let $R$ be a covering surface over $\underline{R}$ and let $L$ be a curve in $R$ determining an accessible boundary point (A.B.P.) $\wp$ with projection $p$. Denote by $V_n(p)$ the neighbourhood of $p$ with diameter $\frac{1}{n}$ and denote by $\mathfrak{B}_n$ the set of $R$ lying over $V_n(p)$, which is composed of at most enumerably infinite number of domains $D^i_n(p)$ $(i=1,2,\dots)$.

*Associated domain.* Let $D^i_n(\wp)$ be a domain, over $V_n(p)$, containing an endpart of $L$. Two arcs $L_1$ and $L_2$ determine the same A.B.P., if and only if, for any number $n$, two associated domains of $L_1$ and $L_2$ are the same. This definition of A.B.P. is clearly equivalent to that of O. Teichmüller. Denote by $n(\underline{z}): \underline{z}\in\underline{R}$ the number of times when $\underline{z}$ is coverd by $R$. Then it is clear that $n(\underline{z})$ is lower semicontinuous. When $\overline{\lim\limits_{z\in\underline{R}}}\, n(\underline{z})>1$, non accessible boundary points are complicated and in our case, it is sufficient to consider only $\mathfrak{A}(R,\underline{R})$, where $\mathfrak{A}(R,\underline{R})$ is the set of all A.B.P.'s.

*Barrier.* Let $B(z): z\in R$ be a function such that $B(z)$ is non negative continuous super-harmonic function and that $\lim\limits_{z\to\wp} B(z)=0$ and moreover for every associated domain $D_m(\wp)$, there exists a number $\delta_m$ with the property that $\inf\limits_{z\notin D_m(\wp)} B(z)>\delta_m(\delta_m>0)$. We call $B(z)$ a barrier at $\wp$. It is well known that $\wp$ is regular for Dirichlet problem of $R$, if and only if, a barrier exists at $\wp$, under the condition that $R$ is a covering surface of $D$-type over $\underline{R}$.

*Lemma. Let $R$ be a covering surface of D-type over $\underline{R}$ and let $\wp$ be an A.B.P. and let $D_n(\wp)$ be an associated domain of $\wp$. We denote by proj $D_n(\wp)$ the projection of $D_n(\wp)$. If proj $\wp$ is regular for proj $D_n(\wp)$, then $\wp$ is regular with respect to $R$.*

In fact, let $B(\text{proj}\ \wp)$ be a barrier of proj $\wp$ with respect to proj $D_n(\wp)$. Then there exists a number $\delta$ such that $B(\text{proj}\ z)>\delta$, when $\underline{z}\notin$ proj $D_m(\wp)$, where $m>n$, for given $D_m(\wp)$. Put $B(z)=\text{Min}(\delta, B(\text{proj}\, z))$ in $D_m(\wp)$ and $B(z)=\delta$ in $R-D_m(\wp)$. Then $B(z)$ is clearly a barrier of $\wp$ with respect to $R$. Thus we have at once the following.

1) See, "Dirichlet problem. III".

*Lemma. Let $\chi$ be a lacunary set which is clearly closed on $\underline{R}$. Then all A.B.P.'s on the boundary of $\chi$ are regular except possibly for a set whose projection is an $F_\sigma$ set of capacity zero.*

In the sequel, let $R$ be a covering surface of finite sheets over $\underline{R}$ such that $n(\underline{z})\leqq N$. It is known that $R$ is a null-or positive boundary Riemann surface according as the set $\underset{\underline{z}}{\mathfrak{E}}\{n(z)\leqq N-1\}:N=\overline{\lim\limits_{z\in \underline{R}}}\, n(\underline{z})$, which is clearly closed, is a set of null-or positive capacity. Let $R$ be a positive boundary Riemann surface. Then we see at once that $R$ is of $F$-type by Theorem 1.1. In such class of Riemann surfaces the following propositions hold.

1) Any A.B.P. is a direct singular point.

Case 1. proj $\mathfrak{p}$ lies on $\underline{R}$. Suppose, $\mathfrak{p}$ is not a direct singular point. Then there exists a connected piece $v$ in $D_n(\mathfrak{p})$ such that $v$ covers proj $\mathfrak{p}$ $S$ times for every number $n$, consequently, there exist discs $c_1, c_2, \ldots\ c_{s'}$ $(S'\leqq S)$ such that $\{c_i\}$ have inner points $\{z_i\}$ whose projections are proj $\mathfrak{p}$. Thus there exists a number $m_0$ such that $(c_i \cap D_m(\mathfrak{p})) : m \geqq m_0$ has no branch points except for $z_i$. It follows that $\{c_i\}$ have no common points each other. Hence $(v\cap D_m(\mathfrak{p}))$: $m\geqq m_0$ does not cover proj $\mathfrak{p}$ $S$ times. This is absurd.
Case 2. If $\mathfrak{p}$ lies on the boundary of $\underline{R}$, our assertion is trivial.

We introduce some definitions as follows:

*Order of an A.B.P. and order of the associated domain $D_n(\mathfrak{p})$.*

The number $\lim\limits_{n\to\infty}$ [sup $n'(\underline{z}) : \underline{z} \in$ proj $D_n(\mathfrak{p})$] and sup [$n'(\underline{z}) : \underline{z} \in$ proj $D_n(\mathfrak{p})$] are called the order of $\mathfrak{p}$ and of $D_n(\mathfrak{p})$ respectively, where $n'(\underline{z})$ is the number when $\underline{z}$ is covered by $D_n(\mathfrak{p})$. We denote the set of $\mathfrak{A}$ $(R, \underline{R})$ of order $n$ by $\mathfrak{A}_n$. The subset of $\mathfrak{A}_n$ such that [sup $n'(\underline{z}) : \underline{z} \in$ proj $D_m(\mathfrak{p})$]$=n$ by $\mathfrak{A}_n^m$ and denote the projection of $\mathfrak{A}_n$ and $\mathfrak{A}_n^m$ by $A_n$ and $A_n^m$ respectively. Then $\mathfrak{A}_n = \bigcup\limits_m \mathfrak{A}_n^m$ and $A_n = \bigcup\limits_m A_n^m$.

2) We easily see that, if sup [$n'(\underline{z}) : \underline{z} \in$ proj $D_m(\mathfrak{p})$]$=n$ and $D_m(\mathfrak{p})$ contains an A.B.P. $\mathfrak{p}'$ of order $n$, then proj $\mathfrak{p}'$ is not contained in proj $D_m(\mathfrak{p})$.

*Theorem 4.1. Let $R$ be a covering surface of finite number of sheets over a null-boundary Riemann surface $\underline{R}$. Then all accessible boundary points are regular for Dirichlet problem except possibly for a subset $\mathfrak{A}^I$ of $\mathfrak{A}(R, \underline{R})$, whose projection is an $F_\sigma$ set of capacity zero.*

Proof. Since the projection of $\mathfrak{A}_N$ is not contained in the projection of $R$, it is clear that all $\mathfrak{A}_N$ is regular except possibly for a subset whose projection is an $F_\sigma$ set of capacity zero. Under the assumption that all points of $\sum\limits_{i=n-p}^{N}\mathfrak{A}_i$ are regular except for a set whose projection is an $F_\sigma$ set of capacity zero, we shall show that $\sum\limits_{i=n-p-1}^{N}\mathfrak{A}_i$ has the property above-mentioned. Suppose, on the contrary, that there exists a set $A^I_{n-p-1}$ such that at least one irregular A.B.P. lies on every point of $A^I_{n-p-1}$. Since $A^I_{n-p-1}=\bigcup_m A^{I,m}_{n-p-1}$, $A_{n-p-1}=\bigcup_m A^m_{n-p-1}$, there exists a number $m$ such that $\operatorname{cap}(A^{I,m}_{n-p-1})>0$. We cover $A^{I,m}_{n-p-1}$ by at most enumerably infinite number of discs $c_1, c_2, \dots$ of diameter $\frac{1}{2m}$, then there is at least one $c'_i$ such that $\operatorname{cap}(c'_i\cap A^{I,m}_{n-p-1})>0$. Let $v_1, v_2, \dots$ be associated domains containing some points of $A^{I,m}_{n-p-1}$ lying over $c'_i$. Then we see that order of $v_i$ is exactly $n-p-1$ and $A^{I,m}_{n-p-1}$ is not contained in the projection of $v_i$ by 1). It follows that every A.B.P. of order $n-p-1$ of $v_i$ is regular except possibly for a set whose projection is a set of capacity zero. This is a contradiction.

Let $p_0$ be a point of $(\underline{R}\cap \underset{\underline{z}}{\mathcal{E}}\{n(\underline{z})=N\})$ which is an open set. Since the projection of irregular accessible boundary points is contained in an $F_\sigma$ set I of capacity zero, we can construct a continuous super-harmonic function $V(\underline{z})$ such that $V(\underline{z})=\infty$ at every point of I and $V(z)$ has one logarithmic singularity at $p_0$ (we can suppose $V(\underline{z})$ is harmonic in the neighbourhood of $p_0$ except one logarithmic singularity). Since $\underline{R}$ is a null-boundary Riemann surface, we can also construct a continuous super-harmonic function $V'(\underline{z})$ which tends to $\infty$ at every boundary point of $\underline{R}$ and has one logarithmic singularity at $p_0$. Put $V''(z)=V(\underline{z})+V'(\underline{z})$, where $z$ lies on $\underline{z}$. Let $z_1, z_2, \dots, z_{N'}$ $(N'\leqq N)$ be all points of $R$ lying on $p_0$ and put $\sum_{i=1}^{N'}2G(z, z_i)+V(z)+C=W(z)$, where $C$ is a suitable constant and $G(z, z_i)$ is the Green's function of $R$ with pole at $z_i$. Then $\operatorname{Min}[1, \varepsilon W(z)]$ is a positive super-harmonic function which tends to $\infty$ on the boundary of $R$ lying on the boundary of $\underline{R}$ and on the irregular accessible boundary points of $R$, for any given positive number $\varepsilon$. By Theorem 1.1 $R$ is a covering surface of Bounded type, whence $R$ is of $D$-type. It follows that $\overline{H}^R_\varphi(z)\geqq\underline{H}^R_\varphi(z)$ for continuous boundary function $\varphi$, where $\overline{H}(\underline{H})$ is lower (upper) envelope of super (sub)-harmonic functions $v(z)$ such that $v(z)\geqq\varphi$ $(\leqq\varphi)$. On the other hand, we can prove the inverse inequality by use of $W(z)$. Hence on such class of Riemann surface the resolutivity of continuous boundary function can be proved by the same method as in the case when $R$ is a subdomain of the $z$-plane.

*Theorem 4.2. Let* $\mathfrak{p}$ *be an A.B.P. whose projection is* $p$ *and let* $G(z, z_0)$ *be the Green's function of* $R$. *If* $\lim_{z\to\mathfrak{p}} G(z, z_0)=0$, $\mathfrak{p}$ *is regular for Dirichlet problem.*

Proof. As we have proved, we can take $n$ so that $p$ is not contained in proj $D_n(\wp)$. Let $C_{\rho_0}$ be the circle $|\underline{z}-p|\leqq\rho_0:\rho_0<\frac{1}{n}$ with respect to the local parameter in the neighbourhood of $p$ and let $D_{\rho_0}(\wp)$ be the associated domain of $\wp$ lying on $C_{\rho_0}$. In the following, we abbreviate $G(z, z_0)$ to $G(z)$ and suppose $G(z)\leqq M$ in $D_{\rho_0}(\wp)$. Put $I_\rho=\{\text{proj } D_{\rho_0}(\wp)\cap \overline{C}_\rho\}:\rho<\rho_0$, where $\overline{C}_\rho$ is the circumference: $|\underline{z}-p|=\rho$. Let $\underline{z}$ be a point of $I_\rho$. Then there exist inner points $z_1, z_2,\dots\ z_{i_0}: i_0\leqq N$ of $D_{\rho_0}(\wp)$. We can choose a number $m_0(\underline{z})$ sucht hat every neighbourhood $V_m(z_i): m>m_0(z_i)$ of $z_i$ which has at most one branch point at $z_i$. Put $\mathfrak{M}\,G(\underline{z})=\underset{i}{\text{Min}}\,G(z_i)$: Then

$$\mathfrak{M}\,G(\underline{z})\geqq\delta(\underline{z})>0.$$

Next we show that $\mathfrak{M}\,G(\underline{z})$ is a measurable function of $\underline{z}$. Let $\{R_\lambda\}$ $(\lambda=1,2,\dots)$ be an exhaustion of $D_{\rho_0}(\wp)$ with compact analytic relative boundary $\{\partial R_\lambda\}$ and denote by $z_1, z_2,\dots z_{s(\lambda)}$ the points such that $z_1, z_2,\dots z_{s(\lambda)}$ are contained in $R_\lambda$ and lie over $\underline{z}$. Since, for every point $\underline{z}\in I_\rho$, there exists $\lambda_0(\underline{z})$ such that $\{z_1, z_2,\dots z_{i_0}\}=\{z_1,\dots z_{s(\lambda)}\}$ for $\lambda\geqq\lambda_0(\underline{z})$. We have $\tilde{G}_\lambda(\underline{z})=\mathfrak{M}\,G(\underline{z})$, where $\tilde{G}_\lambda(\underline{z})=\text{Min}\,[G(z_1),\ G(z_2),\dots\ G(z_{s(\lambda)})]$. Hence

$$\tilde{G}_\lambda(\underline{z})\downarrow\mathfrak{M}\,G(\underline{z}):\underline{z}\in I_\rho,\ \lambda\to\infty.$$

Without loss of generality, we can suppose that $\partial R_\lambda$ does not pass any branch point of $D_{\rho_0}(\wp)$. If $z\notin\partial R_\lambda$, there exist neighbourhoods $V(z_1),\ V(z_2)\cdots V(z_{s(\lambda)})$ such that $V(z_i)$ has no branch point except at most one at $z_i$. Put $\tilde{G}_i(z)=\text{Min}\,[G(z_{i,1}),\ G(z_{i,2}),\ \dots\ G(z_{i,j})]$, where $z_{i,j}$ $(z_{i,j}\in V(z_i))$ has the same projection that of $z_i$. If $z\in(\partial R_\lambda)$, there exists a neighbourhood which has no branch point and has an arc of $\partial R_\lambda$ as its relative boundary. Put $\tilde{G}(z)=M$ if $z\notin V(z_i)$ and $\tilde{G}(z_i)=G(z)$, if $z\in V(z_i)$. Then $\tilde{G}(z_i)$ is upper semi-continuous at $z_i$ and $\tilde{G}_\lambda(z)=\text{Min}\,[\tilde{G}(z_1),\ \tilde{G}(z_2),\dots\ \tilde{G}(z_{i_0})]$ is also upper semi-continuous. Thus $\lim_{\lambda=\infty}\tilde{G}_\lambda(z)=\mathfrak{M}\,G(\underline{z})$ is measurable.

Let $U(z)$ be an upper bounded sub-harmonic function on $D_{\rho_0}(\wp)$ such that $U(z)\leqq|\underline{z}-p|$ on A.B.P.'s and let $H(z)$ be the upper envelope of $\{U(z)\}$. Since $|\underline{z}-p|$ is sub-harmonic, $H(z)\geqq|\underline{z}-p|$. We show that $\lim_{z\to\wp}H(z)=0$. Since $\mathfrak{M}\,G(z)$ is measurable and $\mathfrak{M}\,G(z)>0$ on $I_\rho$, there exists a closed subset $F$ of $I_\rho$ such that mes$|I_\rho-F|<\frac{2\pi\rho}{\rho_0}$ and $\mathfrak{M}\,G(z)\geqq K$ $(K>0)$ on $F$. Introduce the Poisson's integral $I_\rho(z)$ for the circle $|\underline{z}-p|\leqq\rho$ with the value $\rho_0$ on $I_\rho-F$ and zero on the complementary set of $(I_\rho-F)$ with respect to $|\underline{z}-p|=\rho$. Put

$$\varphi(z)=U(z)-\rho-\frac{\rho_0 G(z)}{K}-I_\rho(\underline{z})\quad:\quad\underline{z}=\text{proj } z.$$

Since $\varphi(z)$ is sub-harmonic on $D_{\rho_0}(\mathfrak{p})$, $U(z) \leqq \rho_0$ on A.B.P.'s of $D_{\rho_0}(\mathfrak{p})$ and $U(z)-G(z)<0$ at every inner point of $D_\rho(\mathfrak{p})$ lying on $F$, because $G(z) \geqq \mathfrak{M}\, G(z)$ and $U(z)-I(z) \leqq 0$ at every point lying on $I_\rho - F$. Since $D_\rho(\mathfrak{p})$ is a covering surface of $D$-type, by Theorem 4.1, we have $\varphi(z) \leqq 0$ on $D_\rho(\mathfrak{p})$, whence $H(z) \leqq 2\rho$. Let $\rho \to 0$. Then $\underset{z \to \mathfrak{p}}{H(z)} = 0$.

Put $B(z) = \mathrm{Min}\,(\rho_0, H(z))$, if $z \in D_{\rho_0}(\mathfrak{p})$ and $B(z) = \rho_0$, if $z \in R - D_{\rho_0}(\mathfrak{p})$. Then $B(z)$ is a barrier at $\mathfrak{p}$. Thus $\mathfrak{p}$ is regular.