199. Groups of Isometries of Pseudo-Hermitian Spaces. I

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Recently, Prof. K. Yano [6] has proved beautiful theorems about groups of isometries of n-dimensional Riemannian spaces. We shall study groups of isometries of a pseudo-Hermitian space, by an analogous method.

1. *Preliminary.* Let M be a pseudo-Hermitian space of $2n$ dimensions of class C^4. Then there exists such a tensor field φ_j^i of type (1,1) that

$$\varphi_a^i \varphi_j^a = -\delta_j^i, \qquad g_{ab}\varphi_i^a \varphi_j^b = g_{ij},$$

where g_{ij} is the metric tensor of the space M, and φ_j^i and g_{ij} are of class C^3. If we put

$$\varphi_{ij} = g_{ia}\varphi_j^a,$$

then φ_{ij} is a skew-symmetric tensor by virtue of the relation $\varphi_a^i\varphi_j^a = -\delta_j^i$. When the tensor φ_{ij} is covariant constant, the space M is pseudo-Kählerian.

Let G be a group of isometries of M onto itself and φ_j^i be invariant by G. For brevity, we call G a *group of Hermitian isometries*. If the group G is transitive on M, the space M is called a *homogeneous pseudo-Hermitian space* by definition. Furthermore, if the Riemannian metric g_{ij} of the homogeneous pseudo-Hermitian space M is pseudo-Kählerian, then M is called a *homogeneous pseudo-Kählerian space*.

Let G be a group of Hermitian isometries of a pseudo-Hermitian space M and H the subgroup of G, each transformation of which fixes a given point O of M. That is to say, H is the group of isotropy at the point $O \in M$. Then the subgroup H is isomorphic to a subgroup H' of the unitary group $U(n)$ in n complex variables and H operates on the tangent space of M at the point O in the same manner as the real representation of H' which operates on the $2n$-dimensional real vector space. Throughout this paper, we assume that the group G is always effective on M, and that the group G and the space M are both connected. Moreover, for brevity, it is supposed that the subgroup H of isotropy is connected.

2. *Subgroups of $U(n)$ of dimension $r \geqq n^2-2n+2$.* The following theorem is proved by using the theorems due to D. Montgomery and H. Samelson [3].

THEOREM 1. *Let G be a proper subgroup of the unitary group $U(n)$ in n complex variables. In cases $n \geqq 2$ and $n \neq 4$, if $\dim G \geqq n^2-2n+2$, the group G. is conjugate to $SU(n)$ or P_n. In case $n=4$, it is conjugate to one of the following groups:*

$$SU(4),\ P_4,\ \Lambda_4 \times Sp(2),\ Sp(2).$$

Here, the groups $SU(n)$, $Sp(2)$, P_n and Λ_n are as follows: $SU(n)$ is the unimodular unitary group in n complex variables. $Sp(n)$ is the linear symplectic group in n quaternic variables. P_n and Λ_n are respectively the groups composed of all n-matrices of the types

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad e^{i\theta} I_n,$$

where θ is real, $A \in U(n-1)$ and I_n is the identity n-matrix.

It is easily seen that,

$$\dim U(n)=n^2, \quad \dim SU(n)=n^2-1,$$
$$\dim P_n=n^2-2n+2 \quad \text{and} \quad \dim Sp(2)=10.$$

3. *Dimension of groups of Hermitian isometries.* Let G be a group of Hermitian isometries of a $2n$-dimensional pseudo-Hermitian space. Then it is easily seen that G is of dimension $r \leqq n^2+2n$. The subgroup H of isotropy at a point $O \in M$ is of dimension $r_0 \geqq r-2n$.

If $r=\dim G=n^2+2n-1$, then $r_0 \geqq r-2n=n^2-1$. Hence H is, by virtue of Theorem 1, isomorphic to $U(n)$ or $SU(n)$. Both of the groups $U(n)$ and $SU(n)$ have their natural real representations on the real vector space V^{2n} of $2n$ dimensions, and their real representations have the property of free mobility on V^{2n}. Consequently, H have the property of free mobility at the point $O \in M$. Therefore, *the given group G is transitive on M*, since G and M are connected.

Assume that the group G is of dimension r such that

$$n^2+2n-1 > r > n^2+2 \quad (n \geqq 3).$$

Then we have the following relations:

$$r_0 \geqq r-2n > n^2-2n+2.$$

Hence, if $n \neq 4$, H is isomorphic to $U(n)$ or $SU(n)$ as a consequence of Theorem 1, since $n \geqq 3$. Therefore, it follows that the group G has the property of free mobility at the point O of M. Consequently, the group G is transitive on M. Thus we obtain the relations

$$r = r_0+2n \geqq n^2+2n-1.$$

This inequality contradicts to the given range of r. Hence, *when $n \geqq 3$ and $n \neq 4$, there exists no group of Hermitian isometries of dimension r such that $n^2+2n-1 > r > n^2+2$. When $n=4$ and*

$23=n^2+2n-1>r>n^2+2=18$, *G is transitive and* $\dim G=19$.

Finally, let G be of dimension $r=n^2+2$. Then we have $r_0 \geqq r-2n=n^2-2n+2$. In case $r_0>n^2-2n+2$, G is transitive on M by the same arguments as above, for $n\geqq 2$. Hence $r=r_0-2>n^2+2$ and this contradicts to the assumption that $r=n^2+2$. Therefore, $r_0=n^2-2n+2$ and, consequently, *G is transitive on M* by means of the equality $r=r_0+2n$.

Hence, summing up the results above obtained, we have the following theorem.

THEOREM 2. *Let G be a group of Hermitian isometries of a 2n-dimensional pseudo-Hermitian space M. Then G is transitive on M for* $n\geqq 2$, *if the group G is of dimension* $r\geqq n^2+2$. *In case* $n\geqq 3$ *and* $n\neq 4$ *there exists no group of Hermitian isometries of dimension r such that*

$$n^2+2n-1>r>n^2+2.$$

In the following § 4, we find that the homogeneous pseudo-Hermitian spaces in Theorem 2 are homogeneous Kählerian spaces which are locally symmetric.

4. *Determination of the space M.* Let G/H be a homogeneous pseudo-Hermitian space of $2n$-dimensions and G be of the maximum dimension n^2+2n. Then the subgroup H of isotropy is of dimension n and, consequently, H is isomorphic to $U(n)$. H being compact, the Lie algebra $\mathfrak{g}$ of G is decomposed into a direct sum as a vector space in such a way that

$$\mathfrak{g}=\mathfrak{m}+\mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}]\subset \mathfrak{m},$$

where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to the subgroup H. That is to say, G/H is a reductive homogeneous space [4]. The subalgebra $\mathfrak{h}$ has a decomposition such that $\mathfrak{h}=\mathfrak{h}_1+\mathfrak{h}_2$ and $[\mathfrak{h}_1, \mathfrak{h}_2]=\{0\}$, where $\mathfrak{h}_2$ is isomorphic to the Lie algebra of $SU(n)$ and $\dim \mathfrak{h}_1=1$.

Extending the base field of $\mathfrak{g}$ to the field C of complex numbers, we have a Lie algebra $\mathfrak{g}^c$ over C. Let $\mathfrak{m}^c$, $\mathfrak{h}_1^c$ and $\mathfrak{h}_2^c$ be the subalgebras of $\mathfrak{g}^c$ corresponding respectively to $\mathfrak{m}$, $\mathfrak{h}_1$ and $\mathfrak{h}_2$. Then there exists a decomposition of the vector space $\mathfrak{m}^c$ such that $\mathfrak{m}^c=\mathfrak{m}_1+\overline{\mathfrak{m}}_1$, $\dim \mathfrak{m}_1=\dim \overline{\mathfrak{m}}_1=n$ and, moreover,

$$[U, X]=X \quad \text{for} \quad X\in \mathfrak{m}_1,$$
$$[U, X]=-X \quad \text{for} \quad X\in \overline{\mathfrak{m}}_1,$$

where U is a suitable element of $\mathfrak{h}_1^c$. Therefore, the following relations are easily obtained:

$$[U, [X, Y]]=2[X, Y] \quad \text{for} \quad X, Y\in \mathfrak{m}_1,$$
$$[U, [X, Y]]=-2[X, Y] \quad \text{for} \quad X, Y\in \overline{\mathfrak{m}}_1,$$
$$[U, [X, Y]]=\{0\} \quad \text{for} \quad X\in \mathfrak{m}_1,\ Y\in \overline{\mathfrak{m}}_1.$$

Hence, it follows immediately that

$$[\mathfrak{m}_1, \mathfrak{m}_1]=\{0\}, \qquad [\overline{\mathfrak{m}}_1\overline{\mathfrak{m}}_1]=\{0\},$$

and

$$[\mathfrak{m}_1, \overline{\mathfrak{m}}_1]\subset \mathfrak{h}^c.$$

Thus it follows that $[\mathfrak{m}, \mathfrak{m}]$ is contained in $\mathfrak{h}$, that is, the homogeneous space G/H is locally symmetric [4].

Since g_{ij} and φ^i_j are invariant by G, the tensor φ_{ij} is also invariant by G. Since the homogeneous space G/H is locally symmetric, the tensor φ_{ij} is covariant constant under the canonical connection of G/H, which coincides with the Riemannian connection ([4], Theorem 15.4). Thus G/H is a homogeneous pseudo-Kählerian space. On the other hand, the group G has the property of free mobility on G/H. Hence G/H is a Kählerian space with constant holomorphic sectional curvature.

When G/H has positive holomorphic sectional curvature and G/H is simply connected, G is isomorphic to $SU(n+1)$ and G/H is a complex projective space $P(C, n)$ of n complex dimensions [1].

When G/H has negative holomorphic sectional curvature, G is isomorphic to $S\mathfrak{L}(n+1)$ and G/H is homeomorphic to a Euclidean space E^{2n} of $2n$ dimensions, where $S\mathfrak{L}(n+1)$ is the connected component of the identity in the group composed of all linear transformations of $n+1$ complex variables $(z_1, z_2,\dots, z_{n+1})$ which leave invariant the form

$$z_1\bar{z}_1+z_2\bar{z}_2+\cdots+z_n\bar{z}_n-z_{n+1}\bar{z}_{n+1},$$

and whose determinants are equal to 1 [1].

When G/H is flat, the group G is isomorphic to the group $\mathfrak{M}_H(n)$ of all unitary motions in a unitary space of n complex variables, and moreover G/H is homeomorphic to E^{2n} [1]. Thus we have the following theorem.

THEOREM 3. *Let G/H be a homogeneous pseudo-Hermitian space of dimension $2n$ and $\dim G=n^2+2n$. Then G/H is a homogeneous Kählerian space with constant holomorphic sectional curvature K. When $K>0$ and G/H is simply connected, G is isomorphic to $SU(n+1)$ and G/H is $P(C, n)$. When $K<0$, G is isomorphic to $S\mathfrak{L}(n+1)$ and G/H is homeomorphic to E^{2n}. When $K=0$, G is isomorphic to $\mathfrak{M}_H(n)$ and G/H is homeomorphic to E^{2n}.*

Now, we suppose that G/H is a homogeneous pseudo-Hermitian space and $\dim G=n^2+2n-1$. Then the subgroup H of isotropy is of dimension n^2-1 and, consequently, H is isomorphic to $SU(n)$. Since $SU(n)$ is compact, the homogeneous space G/H is reductive. Then the Lie algebra $\mathfrak{g}$ of G has a decomposition as follows:

$$\mathfrak{g}=\mathfrak{m}+\mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}]\subset \mathfrak{m},$$

where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to the subgroup H.

First, we have easily

$$[\mathfrak{h}, [\mathfrak{m}, \mathfrak{m}]] \subset [\mathfrak{m}, \mathfrak{m}].$$

Consequently, $[\mathfrak{m}, \mathfrak{m}]$ is an invariant subspace with respect to ad $(\mathfrak{h})$: $\mathfrak{g} \to \mathfrak{g}$. Since $\mathfrak{h}$ is simple, one of the following three cases occurs:

$$[\mathfrak{m}, \mathfrak{m}] = \{0\}, \qquad [\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}.$$

In case where $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$, $\mathfrak{m}$ is an ideal and it is easily seen that the radical of $\mathfrak{g}$ is $\{0\}$. Thus $\mathfrak{g}$ is semi-simple in this case [5]. Hence $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. On the other hand, since G is effective, $\mathfrak{h}$ can not be an ideal. Thus the case where $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$ does not occur. Therefore $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ and consequently G/H is locally symmetric.

Since G/H is locally symmetric and G has the property of free mobility, G/H is a homogeneous pseudo-Kählerian space with constant holomorphic sectional curvature. Since the space G/H is locally symmetric, by virtue of a theorem due to K. Nomizu ([4], Theorem 12.1), it is easily seen that the Lie algebra $\mathfrak{l}$ of the restricted holonomy group of G/H is isomorphic to ad $(\mathfrak{h}_1)$: $\mathfrak{m} \to \mathfrak{m}$, where $\mathfrak{h}_1$ is an ideal of $\mathfrak{h}$. Then $\mathfrak{l}$ is a subalgebra of the Lie algebra $\mathfrak{v}$ of $SU(n)$, for ad $(\mathfrak{h})$: $\mathfrak{m} \to \mathfrak{m}$ is the real representation of $\mathfrak{v}$. Thus it follows that $\mathfrak{l}$ is trivial, since $\mathfrak{l} \subset \mathfrak{v}$ and G/H has constant holomorphic sectional curvature. Therefore the given space G/H is flat. Hence the group G is isomorphic to a subgroup $S\mathfrak{M}_H(n)$ of $\mathfrak{M}_H(n)$, where $S\mathfrak{M}_H(n)$ is composed of all elements of $\mathfrak{M}_H(n)$ whose rotation parts belong to $SU(n)$. Moreover, the space G/H is homeomorphic to E^{2n}. Hence we have the following theorem.

THEOREM 4. *Let G/H be a homogeneous pseudo-Hermitian space of dimension $2n$ and $\dim G = n^2 + 2n - 1$. Then G/H is flat and G is isomorphic to $S\mathfrak{M}_H(n)$.*

Theorems 3 and 4 are proved, in [1], by another argument.

By virtue of a theorem in another paper [2], we have the following theorem.

THEOREM 5. *Let G/H be a homogeneous pseudo-Hermitian space of dimension $2n$ and $\dim G = n^2 + 2$. Then, if $n \neq 4$, G/H is locally a product space of a homogeneous pseudo-Kählerian space G_1/H_1 of dimension 2 and another one G_2/H_2 of dimension $2(n-1)$, both of which have constant holomorphic sectional curvature. Moreover, the two groups G and $G_1 \times G_2$ have the same structure.*

It is easily seen that the space G/H in Theorem 5 is a homogeneous space which is locally symmetric.

We obtain at once the following results by analogous arguments as in Theorems 3 and 4. Let G/H be a homogeneous pseudo-Hermitian space of dimension $2n$. Then, *if H is isomorphic to $\Lambda_n \times Sp(n/2)$,*

G/H is flat and is homeomorphic to E^{2n}. Moreover, in this case, G is isomorphic to a subgroup of $\mathfrak{M}_H(n)$ whose rotation part is $\Lambda_n \times Sp(n/2)$. If H is isomorphic to $Sp(n/2)$, we have that G/H is the same as above and G is isomorphic to a subgroup of $\mathfrak{M}_H(n)$ whose rotation part is $Sp(n/2)$.

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