198. Some Properties of Hypernormal Spaces

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E. Hewitt (3) has defined a new class of abstract space called *hypernormal space*. Further results on hypernormal spaces have been obtained by M. Katětov (6). This note is concerned with a consideration of hypernormal spaces.

All spaces considered are Hausdorff or T_2 spaces.

Definition. A space S is called *hypernormal*, if, for any two separated subsets A, B of S, there are two open sets G, H such that $G \supset A$, $H \supset B$ and $\overline{G} \frown \overline{H} = 0$.

We shall first prove the following

Theorem 1. For a Hausdorff space S, the following statements are equivalent.

(1) *S is hypernormal,*

(2) *If A and B are separated, there is a continuous function f on S such that $f(x)=0$ for each $x \in A$ and $f(x)=1$ for each $x \in B$.*

(3) *If A is any subset of S, and f is a bounded continuous function on A, f may be extended to continuous on S.*

In the terminology of E. Čech (1) and E. Hewitt (4), the statement (2) is that any two separated set A, B of S are always *completely separated.*

The statement (3) is essentially due to M. Katětov (6).

Proof. (1) $\rightarrow$ (2)

Let A, B be separated sets of S, then there are two open sets G, H such that $G \supset A$, $H \supset B$ and $\overline{G} \frown \overline{H} = 0$. Since any hypernormal space is normal, there is a continuous function f on S such that $f(x)=0$ on G and $f(x)=1$ on H. Thus A, B are completely separated.

(2) $\rightarrow$ (3)

We can suppose that f on A has the values in $[-1, 1]$. Let $f_0 = f$. We shall define inductively f_n and φ_n. We suppose that f_n are defined. Let

$$A_n = \left\{ x \in S \mid f_n(x) \leq \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n \right\}$$

$$B_n = \left\{ x \in S \mid f_n(x) \geq \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^n \right\}$$

then A_n, B_n are separated. The functions

$$\varphi_n(x)=\frac{1}{3}\left(\frac{2}{3}\right)^n C_{A_n, B_n}(x), \quad f_{n+1}=f_n-\varphi_n$$

are continuous, where the functions $C_{A_n, B_n}(x)$ are continuous on S such that $C_{A_n, B_n}(x)=0$ for $x \in A_n$ and $C_{A_n, B_n}(x)=1$ for $x \in B_n$. Such functions exist by the hypothesis. From $0 \leq C_{A_n, B_n}(x) \leq 1$, we have $0 \leq \varphi_n(x) \leq \frac{1}{3}\left(\frac{2}{3}\right)^n$ and $0 \leq f_n(x) \leq \left(\frac{2}{3}\right)^n$ $(n=0, 1, 2, \ldots)$ for all $x \in S$. Clearly $\sum_{i=1}^{n} \varphi_n(x)$ is uniformly convergent on S. The limit $\varphi(x)$ is continuous on S. $\sum_{i=1}^{n} \varphi_i(x)=f(x)-f_{n+1}(x)$ on A implies $\varphi(x)=f(x)$ on A.

(3) $\to$ (1)

Let A, B be separated. Let $f(x)$ be a function such that $f(x)=0$ on A and $f(x)=1$ on B, then $f(x)$ is continuous on $A \cup B$. From (3), $f(x)$ is extended a continuous function $\varphi(x)$ on S. Let $G=\left\{x \in S \mid \varphi(x)<\frac{1}{3}\right\}$, $H=\left\{x \in S \mid \varphi(x) > \frac{2}{3}\right\}$, then $A \subset G$, $B \subset H$ and $\overline{G} \frown \overline{H}=0$. Therefore Theorem 1 is completely proved.

Let $\beta(S)$ be the Čech compactification of a completely regular space S. Then we have the following

Theorem 2. A space S is hypernormal if and only if, for separated sets A, B in S, the closures of A and B in the space $\beta(S)$ are disjoint.

Proof. If S is hypernormal, then any two separated subsets A, B are completely separated by Theorem 1 (2). Hence Čech theorem implies $\overline{A} \frown \overline{B}=0$ in $\beta(S)$. Conversely, for two separated subsets A, B in S, let $\overline{A} \frown \overline{B}=0$ in $\beta(S)$. Since $\beta(S)$ is normal Hausdorff space, there is a continuous function $f(x)$ such that $f(x)=0$ for $x \in A$ and $f(x)=1$ for $x \in B$. Therefore the hypernormality of S follows directly from Theorem 1 (2). This completes the proof.

Theorem 19 of E. Hewitt (4) and Theorem 1 (2) implies

Theorem 3. A space S is hypernormal if and only if any two separated sets are contained in disjoint Z-sets in S.

For the definition of *Z-set*, see Definition 8 of E. Hewitt ((4), p. 53).

Now we shall consider the Hanner space of two hypernormal spaces. Let X and Y be normal spaces, and B a closed subset of Y, and f: $B \to Y$ a continuous mapping. For the free union of X and Y, we identify every point $y \in B$ with $f(y) \in Y$. Then Z is the identification space which is obtained from X, Y. A topology may be defined on Z by the condition that a set O in Z is open if $j^{-1}(0)$ and $k^{-1}(0)$ are both open, where j is the projection from X to Z, k from Y to Z. The topologized space Z is *Hanner space* of X and Y by f. Such a space was considered by O. Hanner (2) K. Iséki (5).

Theorem 4. The Hanner space Z of hypernormal spaces X and Y by f is hypernormal.

The idea of the proof is essentially due to O. Hanner and the present author ((2) and (5)).

Proof. Let A_1, A_2 be separated sets in Z. Then $A_1 \frown X$, $A_2 \frown X$ are separated in X. Thus there are closure disjoint open sets U_1, U_2 of X such that $U_1 \supset A_1 \frown X$, $U_2 \supset A_2 \frown X$. X is closed in Z, and U_1, U_2 are separated in Z. This implies that $B_1 = A_1 \smile U_1$, $B_2 = A_2 \smile U_2$ are separated in Z. Therefore $k^{-1}(B_1)$, $k^{-1}(B_2)$ are separated in Y. Since Y is hypernormal, there are closure disjoint open sets V_1, V_2 in Y such that $V_1 \supset k^{-1}(B_1)$, $V_2 \supset {}^{-1}(B_2)$. On the other hand, $k \mid Y-B$ is a homeomorphism between $Y-B$ and $Z-X$. Therefore $G_1 = k(V_1 - B) \smile U_1$, $G_2 = k(V_2 - B) \smile U_2$ are closure disjoint. That the two sets G_1, G_2 are open is proved by a method similar to one proving the previous theorem. (See for detail, O. Hanner (2) or K. Iséki (5).) The proof is complete.

A hypernormal space X is called an *AR* (*ANR*) *for the hypernormal class* whenever a topological imbedding of X as a closed subset X_1 of every hypernormal space Y is a retract (neighborhood) of Y.

Then the following theorems are an easy consequence of Theorem 4 and their proofs are similar to the previous ones (K. Iséki (5), p. 145), therefore we shall omit the details.

Theorem 5. A hypernormal space X is an AR for hypernormal class if and only if any continuous mapping f: $B \to X$ of a closed subset of a hypernormal space Y can be extended to Y.

Theorem 6. A hypernormal space X is an ANR for hypernormal class if and only if, for any mapping f: $B \to X$ of a closed subset of a hypernormal space Y, there is an extension F: $U \to X$ of f to a neighborhood U of B in Y.

References

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