195. A Simple Proof of Littlewood's Tauberian Theorem

By Shin-ichi IZUMI

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Littlewood's tauberian theorem[1] reads as follows:

If $\sum_{n=0}^{\infty} a_n x^n$ *converges for* $|x|<1$ *and*

$$\lim_{x\to 1}\sum_{n=0}^{\infty} a_n x^n = 0, \tag{1}$$

$$|a_n| \leqq A/n \qquad (n=1,\ 2,\ \ldots), \tag{2}$$

then $\sum_{n=0}^{\infty} a_n = 0$.

Two simple proofs of this theorem were given by J. Karamata.[2] We shall give another simple one.

Let

$$a(t)=a_n \qquad (n \leqq t < n+1,\ n=1,\ 2,\ \ldots),$$

$$S(t)=\int_0^t a(u)du,$$

then[3]

$$f(s)=\int_0^\infty a(t)e^{-st}dt=\sum_{n=0}^{\infty} a_n e^{-st}\int_0^1 e^{-st}dt.$$

Hence the conditions (1) and (2) become

$$|a(t)| \leqq A/t,\ \lim_{s\to 0} f(s)=0. \tag{3}$$

We shall further put

$$P_u(t)=e^{-ut}\sum_{m<u}\frac{(ut)^m}{m!},$$

$$g(t)=1\ (0\leqq t<1), \qquad g(t)=0\ (t>1).$$

Then we have

$$S(T)=\int_0^T a(t)dt=T\int_0^1 a(Tt)dt=T\int_0^\infty a(Tt)g(t)dt$$

$$=T\int_0^\infty a(Tt)[g(t)-P_u(t)]dt+T\int_0^\infty a(Tt)P_u(t)dt$$

$$=S_1(T)+S_2(T),$$

say. By (3)

$$|S_1(T)| \leqq A\int_0^\infty \frac{1}{t}\,|g(t)-P_u(t)|\,dt$$

1) J. E. Littlewood: Proc. London Math. Soc., **9** (1918).

2) J. Karamata: Math. Zeits., **32** (1932); **56** (1952). Cf. K. Knopf: Theorie der unendlichen Reihen, 2te Aufl. R. Wielandt has given a simple proof of the one side theorem due to G. H. Hardy and J. E. Littlewood in Math. Zeits., **56** (1952).

3) This is the form used by A. Korevaar in his papers in Indak. Math. (1953–1954).

$$\leqq A\int_0^1 \frac{1}{t}\left|e^{-ut}\sum_{m<u}\frac{(ut)^m}{m!}-1\right|dt-A\int_1^\infty \frac{e^{-ut}}{t}\Big(\sum_{m<u}\frac{(ut)^m}{m!}\Big)dt$$
$$=AS_{1,1}-AS_{1,2},$$

say. We have

$$S_{1,1}=\int_0^1 \frac{e^{-ut}}{t}\Big(\sum_{m>u}\frac{(ut)^m}{m!}\Big)dt$$
$$=\sum_{m>u}\frac{1}{m!}\int_0^1 (ut)^{m-1}e^{-mt}u\,dt=\sum_{m>u}\frac{1}{m!}\int_0^u t^{m-1}e^{-t}dt$$
$$=\sum_{u<m\leqq au}+\sum_{au<m}=S_{1,1,1}+S_{1,1,2},$$

say, where $a>1$. Since

$$\frac{1}{(m+1)!}\int_0^u t^m e^{-t}dt<\frac{u}{m+1}\,\frac{1}{m!}\int_0^u t^{m-1}e^{-t}dt,$$

we get

$$S_{1,1,2}\leqq \frac{1}{(au)!}\int_0^u t^{au-1}e^{-t}dt\Big[1+\frac{u}{au+1}+\frac{u^2}{(au+1)(au+2)}+\cdots\Big]$$
$$\leqq A\frac{(au-1)!}{(au)!}\,\frac{1}{1-1/a}\leqq A/(a-1)u,$$

and also

$$S_{1,1,1}\leqq \sum_{u<n\leqq au}\frac{1}{m}\leqq A\ \log a\leqq A(a-1).$$

If we take $a=1+1/\sqrt{u}$, then $S_{1,1}\leqq A/\sqrt{u}$.

Similarly we have

$$S_{1,2}=\sum_{m<u}\frac{1}{m!}\int_0^\infty (ut)^{m-1}e^{-ut}u\,dt=\sum_{m<u}\frac{1}{m!}\int_0^\infty t^{m-1}e^{-t}dt$$
$$=\int_u^\infty e^{-t}\sum_{m<u}\frac{t^{m-1}}{m!}dt=\int_u^{(1+\varepsilon)u}+\int_{(1+\varepsilon)u}^\infty=S_{1,2,1}+S_{1,2,2},$$

say. Evidently $S_{1,2,1}\leqq\varepsilon$ and

$$S_{1,2,2}\leqq \frac{e^{-\varepsilon u}}{(u-1)!}\int_{(1+\varepsilon)u}^\infty t^{u-1}e^{-t/(1+\varepsilon)}dt$$
$$=\frac{e^{-\varepsilon u}}{(u-1)!}(1+\varepsilon)^u\int_u^\infty t^{u-1}e^{-t}dt\leqq((1+\varepsilon)e^{-\varepsilon})^u.$$

If we take $\varepsilon=1/\sqrt[3]{u}$, then $S_{1,2}=S_{1,2,1}+S_{1,2,2}\leqq A/\sqrt[3]{u}$.

Hence $|S_1(T)|\leqq A/\sqrt[3]{u}$.

On the other hand, since $f(s)=o(1)$ and $f^{(m)}(s)=O(s^{-m})$ $(m=1,2,\cdots)$ as $s\to 0$, imply $f^{(m)}(s)=o(s^{-m})$,[4] we get

$$S_2(T)=T\int_0^\infty a(Tt)P_u(t)dt=\int_0^\infty a(t)P_u(t/T)dt$$

4) J. E. Littlewood: Loc. cit.

$$= \sum_{m<u} \frac{u^m}{m!} \int_0^\infty a(t) \frac{t^m}{T^m} e^{-ut/T} dt$$

$$= \sum_{m<u} \frac{1}{m!} \left(\frac{u}{T}\right)^m f^m\left(\frac{u}{T}\right) = o(u),$$

where o may be taken $1/u^2$ for large T. Thus we have $S_2(T) = o(1)$ as $T \to \infty$. The theorem is now completely proved.