6. *Dirichlet Problem on Riemann Surfaces.* V
(*On Covering Surfaces*)

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1) Covering Surfaces over a Null-Boundary Riemann Surface

Let $\underline{R}$ be a null-boundary Riemann surface and let R be a covering surface of F-type over $\underline{R}$. We denote by $\mathfrak{A}(R, \underline{R}^*)$ the set of all A.B.P.'s of R. Let $\mathfrak{F}$ be a closed set of $\mathfrak{A}(R, \underline{R}^*)$. The upper class $U^R_{\mathfrak{F}}$ is the set of all non negative continuous super-harmonic functions $U(z)$ such that $\lim U(z) \geqq 1$ along every curve tending to $\mathfrak{F}$. We denote by $\overline{H}^R_{\mathfrak{F}}(z)$ the lower envelope of $U^R_{\mathfrak{F}}$. Similarly the lower class $B^R_{\mathfrak{F}}$ is the class of all bounded continuous sub-harmonic functions $V(z)$ such that $\lim V(z) \leqq 0$ along every curve tending to the boundary except $\mathfrak{F}$. Further it is clear that $\overline{H}^R_{\mathfrak{F}}(z) \geqq \underline{H}^R_{\mathfrak{F}}(z)$ on a covering surface of D-type. If they coincide at one point of R. Then they are identical.

Lemma. Let $\mathfrak{F}$ be a closed set of $\mathfrak{A}(R, \underline{R}^*)$ of a covering surface of F-type. Then

$$\overline{H}^R_{\mathfrak{F}}(z) = \underline{H}^R_{\mathfrak{F}}(z).$$

Proof. We map the universal covering surface R^∞ of R onto the unit circle $U_\xi : |\xi| < 1$. Since by assumption, the mapping function $f(\xi) : R^\infty \to R + \mathfrak{A}(R, \underline{R}^*)$ has angular limits almost everywhere on $|\xi| = 1$ and since $\mu(R, B) = 0$, where $\mu(R, B)$ is the outer harmonic measure of the boundary of R lying on the boundary of $\underline{R}$. We can suppose that $f(\xi)$ has angular limits lying in $\underline{R}$. Let $\mathfrak{F}'_n$ be the set of points of $R + \mathfrak{A}(R, \underline{R}^*)$ which have distance $\leqq \frac{1}{n}$ from $\mathfrak{F}$. Put $\mathfrak{F}_n = \mathfrak{F}'_n \cap \mathfrak{A}(R, \underline{R}^*)$ and let F be the regular image[1] of $\mathfrak{F}$. Then, since R is a covering surface of F-type, F_n is measurable and $f(\xi)$ has angular limits at F'_n, where $\text{mes}\,|F_n - F'_n| = 0$. Thus we can suppose $f(\xi)$ has angular limits at F_n. Let $\{R_m\}$ be an exhaustion of R with compact relative boundaries $\{\partial R_m\}$ and let $\partial \mathfrak{F}'_n$ be the relative boundary of $\mathfrak{F}'_n$. Let $\omega^n_{m,m+i}(z)$ be a harmonic function in $R_{m+i} - (\mathfrak{F}'_n \cap (R_{m+i} - R_m))$ such that $\omega^n_{m,m+i}(z) = 1$ on $\partial(\mathfrak{F}'_n \cap (R_{m+i} - R_m))$ and $\omega^n_{m,m+i}(z) = 0$ on $\partial R_{m+i} - \mathfrak{F}'_n$. Then we see that $\omega^n_{m,m+i}(z) \uparrow \omega^n_m(z)$ and $\omega^n_m(z) \downarrow \omega^n(z)$. Thus $\omega^n(z)$ is contained in the class $U^R_{\mathfrak{F}}$ for

1) See, "Dirichlet problem. III".

every n and $\omega(z)=\lim_n \omega^n(z)\geqq \overline{H}^R_{\mathfrak{F}}(z)$. On the other hand $f(\xi)$ has angular limits not contained in $\mathfrak{F}$ almost everywhere on CF, where $\lim_n F_n=F$ and CF is the complementary set of F with respect to $|\xi|=1$. We show $\omega(z)$ has angular limit zero almost everywhere on CF. If it were not so, we can find a closed set E of positive measure in CF_n, because $\lim_n |\mathrm{mes}\,|F_n-F||=0$, such that $\omega(z)$ has an angular limit $\delta(\delta>0)$ on E. Let D_E be a domain containing an angular end-part: $|\arg(1-e^{-i\theta}\xi)|<\frac{\pi}{2}-\delta_0$ at every point ξ of E and let $C(r)$ be a circle such that $|\xi|<r$. Since the mapping function has angular limits contained in $\underline{R}$ almost everywhere on CF_n, $f(\xi)$ tend to A.B.P.'s outside of $\mathfrak{J}_n$ in angular domain. Hence there exists r, n_0 and a closed set $E'(\subset E)$ such that, if $\xi\in(D_{E'}\cap(U_\xi-C(r))$, then $f(\xi)\notin\mathfrak{F}'_n(n\geqq n_0)$. As to the mapping $R^\infty\to U_\xi$, R^∞_m is mapped onto the simply connected domain containing $\xi=0$. Hence the image of $\partial\mathfrak{F}'_n(n>n_0)$ does not fall in $D_{E'}\cap(U_\xi-C(r))$. Since $D_{E'}\cap(U_\xi-C(r))$ is composed of at most a finite number of domains, there exists at least one component $D'_{E'}$ with the property that $D'_{E'}$ has a set on $|\xi|=1$ of positive measure as its boundary. Since the boundary of $D'_{E'}$ is rectifiable, there exists a harmonic function $\hat{\omega}(\xi)$ such that $\hat{\omega}(\xi)=1$ on the boundary of $D'_{E'}$ except one lying on $|\xi|=1$ and $\hat{\omega}(\xi)=0$ on the boundary of $D'_{E'}$ lying on $|\xi|=1$. Consider $\omega^n_{m,m+i}(z)$ on $D'_{E'}$. Then we see by the maximum principle that $\omega^n_{m,m+i}(z)\geqq\hat{\omega}(\xi)$. Let $i\to\infty$ and then $m,n\to\infty$. We have $\omega(z)\geqq\hat{\omega}(\xi)$. This contradicts that $\omega(z)$ has an angular limit zero. Hence $\omega(z)=0$ almost everywhere on CF. Let $\omega(U_\xi,F)$ be the harmonic measure of F. Then we have $\omega(U_\xi,F)=\overline{H}^R_{\mathfrak{F}}(z)$, and the inverse inequality is clear.

Put $\Omega_n=R-\mathfrak{F}'_n$ and let J^λ be the domain where $\overline{H}^R_{\mathfrak{F}}(z)>\lambda$ and put $T^{\lambda,n}=\Omega_n\cap J^\lambda$. Let $\omega^{\lambda,n}_{m,m+i}(z)$ be a harmonic function in $R_{m+i}-((R_{m+i}-R_m)\cap T^\lambda_n)$ such that $\omega^{\lambda,n}_{m,m+i}(z)=1$ on $\partial(R_{m+i}-R_m)\cap T^{\lambda,n})$ and $\omega^{\lambda,n}_{m,m+i}(z)=0$ on $\partial R_{m+i}-T^{\lambda,n}$. Then $\omega^{\lambda,n}_{m,m+i}(z)\uparrow\omega^{n,\lambda}_m(z)$ and $\omega^{n,\lambda}_m(z)\downarrow\omega^{n,\lambda}(z)$. As above, we can prove that $\omega^{\lambda,n}(z)$ has angular limit zero almost everywhere on F_{2n}. On the other hand since $\omega^{\lambda,n}(z)<\frac{1}{\lambda}\overline{H}^R_{\mathfrak{F}}(z)$, $\omega^{\lambda,n}(z)$ has angular limit zero on CF_n. Therefore $\omega^{\lambda,n}(z)\equiv 0$. Thus we can easily construct a super-harmonic function $W(z)$ such that $W(z)=\infty$ at every point of the boundary defined by a non compact domain $\sum_{\lambda>0}\sum_{n=1}^{\infty}T^{\lambda,n}$, because every $T^{\lambda,n}$ determines a set of outer harmonic measure zero.[2] Then it is clear that $\overline{H}^R_{\mathfrak{F}}(z)-\varepsilon W(z)$ is contained in $B^R_{\mathfrak{F}}$ class for every positive number ε. Hence $\overline{H}^R_{\mathfrak{F}}(z)\leqq\underline{H}^R_{\mathfrak{F}}(z)$. Thus

2) See, "Harmonic measures and capacity. I".

$\overline{H}^R_{\mathfrak{F}}(z)=\underline{H}^R_{\mathfrak{F}}(z)=\omega(U,F)$. Similarly, for open set Ω of $\mathfrak{A}(R,\underline{R})$, we have $\overline{H}^R_{\Omega}(z)=\underline{H}^R_{\Omega}(z)$.

Let $\varphi(\mathfrak{p}):\mathfrak{p}\in\mathfrak{A}(R,\underline{R}^*)$ be a real valued function. Define the upper class U^R_φ consisting of all the lower bounded continuous superharmonic functions such that $\lim u(z)\geqq\varphi(\mathfrak{p})$ and let its lower envelope by $\overline{H}^R_\varphi(z)$. We also define the lower class and its upper envelope $\underline{H}^R_\varphi(z)$. If $\overline{H}^R_\varphi(z)=\underline{H}^R_\varphi(z)$ holds, then $\varphi(\mathfrak{p})$ is called a resolutive boundary function and we denote the common envelope by $H^R_\varphi(z)$.

Theorem 5.1. Let R be a covering surface of F-type and let $\varphi(\mathfrak{p})$ be semi-continuous function. Then $\varphi(\mathfrak{p})$ is resolutive.

Proof. Let $\varphi(\mathfrak{p}):M\geqq\varphi(\mathfrak{p})\geqq m$ be an upper (lower) semi-continuous function on $\mathfrak{A}(R,\underline{R}^*)$ and divide the interval $[m,M]$ into sub-intervals such that $m=c_0<c_1,\dots,c_n=M$, and $c_{i+1}-c_i=\frac{M-m}{n}$. Denote the set $\underset{\mathfrak{p}}{\mathcal{E}}\{\varphi(\mathfrak{p})\geqq c_i\}$ by A_i which is closed and $\underset{\mathfrak{p}}{\mathcal{E}}\{c_{i+1}\geqq\varphi(\mathfrak{p})>c_i\}$ by E_i respectively. Let U_{A_i} and B_{B_i} be the upper and the lower classes of the characteristic function of A_i. Put $U_{E_i}(z)=U_{A_i}(z)-V_{A_{i+1}}(z)$ $(V_{E_i}(z)=V_{A_i}(z)-U_{A_{i+1}}(z))$, where $U_{A_i}(z)(V_{A_i}(z))$ is a function contained in the class $U_{A_i}(B_{A_i})$. Then $U_{E_i}(z)(V_{E_i}(z))$ is super(sub)-harmonic and $\lim\limits_{z\to E_i}U_{E_i}(z)=1$ $(\lim\limits_{z\to CE_i}V_{E_i}(z)\leqq 0)$. Thus $U_{E_i}(z)(V_{E_i}(z))$ is contained in the class $U_{E_i}(B_{E_i})$. Hence $U^n_\varphi(z)=\sum\limits_{i=0}^{n-1}c_{i+1}U_{E_i}(z)\,(V^n_\varphi(z)=\sum\limits_{i=1}^{n}c_iV_{E_i}(z))$ is contained in the class $U^R_\varphi(B^R_\varphi)$ but the lower envelope $\overline{H}^R_{E_i}(z)(\underline{H}^R_{E_i}(z))$ of $U_{E_i}(B_{E_i})$ is equal to the harmonic measure of the image of E_i on $|\xi|=1$. Thus we have

$$(U^n_\varphi(z)-V^n_\varphi(z))\leqq\frac{1}{n}\sum_{i=0}^{n-1}\omega(U_\xi,E_i),\ \text{let } n\to\infty.\ \text{Then}$$

$$\overline{H}^R_\varphi(z)=\underline{H}^R_\varphi(z)=\int\varphi\,d\mu,$$

where μ is the harmonic measure.

From the general theory of Dirichlet problem, we have next

Lemma. $H^R_\varphi(z)$ is the upper envelope of $H^R_\Psi(z)$, where $\psi<\varphi$ and ψ is upper bounded and semi-continuous on $\mathfrak{A}(R,\underline{R}^*)$. The similar fact holds for $\overline{H}^R_\varphi(z)$.

In the same manner used by M. Brelot, we have

Theorem 5.2. In order that $\varphi(\mathfrak{p})$ is resolutive, it is necessary and sufficient that $\varphi(\mathfrak{p})$ is integrable in the narrow sense.

2) Covering Surfaces over a Positive Boundary Riemann Surface

Let $\underline{R}$ be a positive boundary Riemann surface, $G(\underline{z},\underline{z}_0)$ be the Green's function and let $h(\underline{z})$ be its conjugate. Put $k(\underline{z})=e^{-G-ih}$. We define the length of a curve $\underline{L}$ by $\int_{\underline{L}}dk(z)$ and the distance between two points $\underline{z}_1$ and $\underline{z}_2$ by the lower limit of the lengths of all curves $\underline{L}$ connecting $\underline{z}_1$ with $\underline{z}_2$ in $\underline{R}$. Then we have a metric space $\underline{R}^*$ by completion of this metric. Let R be a covering surface over $\underline{R}$. The distance between two points z_1 and z_2 of R is defined by the diameter of the projection of all curves connecting z_1 with z_2 in R. We map the universal covering surface R^∞ of R onto the unit circle $U_\xi: |\xi|<1$ and let $f(\xi)$ be the mapping function from R^∞ to $\underline{R}^*$. If a curve l_z on R tends to the boundary of R and its projection tends to a point $\underline{R}^*$, we say that l_z determines an A.B.P. which means that the composed function $R^\infty\to\underline{R}\to k(z)=w$, has limit w_0 on the w-plane along the image l_ξ of l_z on R^∞. Since $|k(z)|<1$, the image l_ξ tends to a point on $|\xi|=1$ where $U_\xi\to w$-plane has an angular limit w_0, whence $U_\xi\to\underline{R}^*$ has an angular limit. Since $f(\xi)\to w$ is bounded, it follows that $R\to\underline{R}^*$ has angular limits almost everywhere on $|\xi|=1$. Thus R is a covering surface of F-type. Let $\mathfrak{A}_\xi$ be the image of $\mathfrak{A}(R,\underline{R}^*)$ and F be the image of a closed set $\mathfrak{F}$ of $\mathfrak{A}(R,\underline{R}^*)$. Then we see that they are measurable. Hence we have the same results about the Dirichlet problem as in the case when $\underline{R}$ is a null-boundary Riemann surface.

3) Covering Surface of Finite Number of Sheets

Let $\underline{R}$ be a positive boundary Riemann surface. We have introduced A-topology on an abstract Riemann surface $\underline{R}$. Then $\underline{R}$ is a metric space $\underline{R}^*$. In this topology, every boundary component is one point. We define the length of a continuous curve $\underline{L}$ by $\overline{\lim}\sum_{i=1}^{n}|\underline{z}_i, \underline{z}_{i+1}|$, where $\{\underline{z}_i\}$ are points of $\underline{L}$ and $|\underline{z}_i,\underline{z}_{i+1}|$ is the distance between $\underline{z}_i$ and $\underline{z}_{i+1}$ in A-topology. We introduce another topology as follows: If a curve $\underline{L}$ tends to a point $\underline{p}$ of $\underline{R}^*$ and the Green's function $G(\underline{z},\underline{z}_0)$ has a limit γ_1 when $\underline{z}$ tends to $\underline{p}$ along $\underline{L}$. We say that $\underline{L}$ determines a point $\underline{z}$ of $\underline{R}^{**}$. Let $G_{\delta_1}(\underline{z}_1)$ be the domain of $\underline{R}$ in which $\gamma_1+\delta_1>G(\underline{z},z_0)>\gamma_1-\delta_1$. Let $\underline{z}_1$ and $\underline{z}_2$ be two points of $\underline{R}^{**}$. Then $G_{\delta_1}(\underline{z}_1)\ni\underline{z}_2$ for sufficiently large number δ_1. Connect $\underline{z}_1$ and $\underline{z}_2$ by a curve $\underline{L}$ in $G_{\delta_1}(\underline{z}_1)$ and denote by $\Lambda(\underline{L})$ the diameter of $\underline{L}$. Put $\delta_2=\inf\Lambda(\underline{L})$ of all curves above-mentioned. It is clear that δ_2 increases when δ_1 decreases. We define the distance between $\underline{z}_1$ and $\underline{z}_2$ by $\inf(\delta_1+\delta_2)$. The topology by this metric will be called B-topology and $\underline{R}^{**}$ is also a metric space.

It is clear that B-topology is the same as the original topology when $\underline{R}^{**}$ is restricted in $\underline{R}$, because $G(\underline{z},\underline{z}_0)$ is continuous in $\underline{R}$. Let R be a covering surface over $\underline{R}$. If a continuous curve L converges to the boundary of R and its projection tends to a point of $\underline{R}^{**}$, we say that L determines an A.B.P. The distance between z_1 and z_2 is defined as usual by the lower limit of the diameter in

B-topology of the projection of all curves connecting z_1 and z_2 in R. Thus we have a metric space $R^{**}=R+\mathfrak{A}(R,\underline{R}^{**})$ from R by the completion of this metric, where $\mathfrak{A}(R,\underline{R}^{**})$ is a set of all A.B.P.'s of R.

Theorem 5.3. Let R be a covering surface of finite number of sheets over $\underline{R}$. Then A.B.P.'s are regular for Dirichlet problem except possibly for a set whose projection is a set of capacity zero.

Proof. Let $\wp$ be an A.B.P. of R^{**} whose projection is $\underline{z}'$ of $\underline{R}^{**}$ such that $\underset{z\to\underline{z}'}{G}(\underline{z},\underline{z}_0)=0$ in B-topology. Denote by $V_{2n}(\underline{z}')$ a neighbourhood with radius $\frac{1}{n}$ in A-topology and denote by $G_{2n}(\underline{z}')$ a domain in which $G(\underline{z},\underline{z}^0)<\frac{1}{2n}$. We can find a neighbourhood $V'(\underline{z}')$ in $V_{2n}(\underline{z}')$ with a compact relative boundary $\partial V'(\underline{z}')$. Hence $\underset{z\in\partial V'}{\mathrm{Min}}(G(z,z))\geqq\delta>0$. Put $\mathrm{Min}[\delta, G(\underline{z},\underline{z}_0)]=B_n(\underline{z})$ in $V'(\underline{z})$ and $B_n(\underline{z})=\delta$ in $\underline{R}^{**}-V'(\underline{z}')$. Let $V_n^*(z')$ be a neighbourhood of $\underline{z}'$ in B-topology with radius $\frac{1}{n}$, and let $V'_{2n}(\underline{z}')$ be a component of $(G_{2n}\cap V_{2n}(z')$ containing an end-part of the curve determining $\underline{z}'$. Then clearly $V'_{2n}(\underline{z}')\subset V_n^*(\underline{z}')$ and $G(\underline{z},z_0)\geqq\mathrm{Min}\left[\delta,\frac{1}{2n}\right]=\delta_{2n}$ on the boundary of $(G_{2n}\cap V_{2n}(z'))$, whence $B_n(z)\geqq\delta'(>0)$ in $(\underline{R}^{**}-V_n^*(\underline{z}'))$. On the other hand $G(\underline{z},z_0)$ tends to zero, when $\underline{z}$ tend to $\underline{z}'$. Put $B(z)=\sum\frac{B_n(\underline{z})}{2^n\delta'_{2n}}$. Then $B(\underline{z})$ is a barrier at $\underline{z}'$. Put $B(z)=B(\underline{z})$ on R. Then $B(z)$ is a barrier at $\wp$. Let $\{R_m\}$ be an exhaustion of R and let A_m be a set of A.B.P.'s whose projection are contained in R_m. We can prove as the previous[3] that A_m are regular except possibly for a set whose projection is a set I_m of capacity zero. On the other hand the set B_λ of ideal boundary defined by a non compact domain D_λ of $\underline{R}$ in which $G(\underline{z},\underline{z}_0)>\lambda(\lambda>0)$ is a set of capacity zero.[4] Therefore the projection of irregular A.B.P.'s are contained in B_λ or $\sum_{m=1}^{\infty}I_m$. Hence we have the theorem.

Since R is a covering surface of F-type over $\underline{R}^{**}$. We can prove also $\overline{H}_\varphi^R(z)=\underline{H}_\varphi^R(z)$ for a semi-continuous function on $\mathfrak{A}(R,\underline{R}^{**})$ by the same method in the previous.

The topology in 2) is more precise than that of 3) but at present we cannot introduce the notion of regular point on this topology.

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3) See, "Dirichlet problem. IV".
4) See, "Harmonic measures and capacity. II".