43. *Probabilities on Inheritance in Consanguineous Families. XII*

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IX. Combinations through extreme consanguineous marriages, 1

1. Mother-descendants combinations

The main purpose of the present and the subsequent chapters is to deal with a mother-descendants combination through repeated extreme consanguineous marriages, of which the reduced probability is designated by

$$\kappa_{(\mu\nu;0)_t|\mu\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)\equiv\kappa_{\mu_1\nu_1;0|\cdots|\mu_t\nu_t;0|\mu_{t+1}\nu_{t+1}}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2),$$

and also several related combinations.[1] Among such a combination the most extreme case $\mu_r=\nu_r=1$ for $1\leqq r\leqq t$ has been observed already in the preceding chapter. For the remaining cases we shall restrict ourselves to consider both canonical cases $\mu_r=1<\nu_r$ and $\mu_r,\,\nu_r>1$ for $1\leqq r\leqq t$, so that some intermediate cases will be omitted.

We first consider the probability $\kappa_{(\mu\nu;0)_t|\mu\nu}$ for the generic case, namely with $\mu_r,\,\nu_r>1$ for $1\leqq r\leqq t$. The reduced probability is evidently given by a recurrence equation

$$\kappa_{(\mu\nu;0)_t|\mu\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\sum\kappa_{(\mu\nu;0)_{t-1}|\mu_t\nu_t}(\alpha\beta;\,ab,cd)\varepsilon_\mu(ab,cd;\,\xi_1\eta_1)\varepsilon_\nu(ab,cd;\,\xi_2\eta_2)$$

where the summation extends over all the possible pairs of genotypes A_{ab} and A_{cd}.

Under the initial condition expressed by

$$\kappa_{\mu\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\bar{A}_{\xi_1\eta_1}\bar{A}_{\xi_2\eta_2}+2^{-\mu+1}\bar{A}_{\xi_2\eta_2}Q(\alpha\beta;\,\xi_1\eta_1)+2^{-\nu+1}\bar{A}_{\xi_1\eta_1}Q(\alpha\beta;\,\xi_2\eta_2)$$
$$+2^{-\lambda}T(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)\qquad(\lambda=\mu+\nu-1),\text{[2]}$$

it can be shown that *the recurrence equation is solved by*

$$\kappa_{(\mu\nu;0)_t|\mu\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\bar{A}_{\xi_1\eta_1}\bar{A}_{\xi_2\eta_2}$$
$$+\Lambda_t\{2^{-\mu+1}\bar{A}_{\xi_2\eta_2}Q(\alpha\beta;\,\xi_1\eta_1)+2^{-\nu+1}\bar{A}_{\xi_1\eta_1}Q(\alpha\beta;\,\xi_2\eta_2)\}$$
$$+x_{t+1}\bar{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\,\xi_2\eta_2)+y_{t+1}S(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)+z_{t+1}T(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2),$$

where the coefficients are defined by

$$\Lambda_t=\prod_{r=1}^{t}(2^{-\mu_r}+2^{-\nu_r}),\qquad\lambda_r=\mu_r+\nu_r-1,$$

1) Previous papers under the same title have been published in Proc. Japan Acad. **30** (1954), 42-52, 148-155, 236-247, 636-654. For full details, cf. Y. Komatu and H. Nishimiya, Probabilistic investigations on inheritance in consanguineous families. Bull. Tokyo Inst. Tech. (1954), 1-66, 67-152, 153-222 et seq.

2) With respect to several quantities involved in the formula, cf. the papers cited in 1).

$$x_{t+1}=2\sum_{r=1}^{t}\prod_{s=r+1}^{t+1}2^{-\lambda_s},\quad y_{t+1}=2\sum_{r=1}^{t}A_r\prod_{s=r+1}^{t+1}2^{-\lambda_s},\quad z_{t+1}=\prod_{s=1}^{t+1}2^{-\lambda_s}.$$

We supplement here two further special cases which will be availed later.

First, we consider the combination $(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)_{(\mu\nu;\,0)_t|11}$ with $\mu_r,\ \nu_r>1$ for $1\leqq r\leqq t$. The defining equation of its reduced probability is given by

$$\kappa_{(\mu\nu;\,0)_t|11}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\sum\kappa_{(\mu\nu;\,0)_{t-1}|\mu_t\nu_t}(\alpha\beta;\ ab,\ cd)\varepsilon(ab,cd;\ \xi_1\eta_1)\varepsilon(ab,cd;\ \xi_2\eta_2).$$

We introduce three quantities defined by

$$\mathfrak{R}(\xi_1\eta_1,\ \xi_2\eta_2)=\sum\bar{A}_{ab}Q(ab;\ cd)\varepsilon(ab,\ cd;\ \xi_1\eta_1)\varepsilon(ab,\ cd;\ \xi_2\eta_2),$$
$$\mathfrak{S}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\sum S(\alpha\beta;\ ab,\ cd)\varepsilon(ab,\ cd;\ \xi_1\eta_1)\varepsilon(ab,\ cd;\ \xi_2\eta_2),$$
$$\mathfrak{T}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\sum T(\alpha\beta;\ ab,\ cd)\varepsilon(ab,\ cd;\ \xi_1\eta_1)\varepsilon(ab,\ cd;\ \xi_2\eta_2).$$

Their values may be obtained by means of actual computation, while they will be omitted here for economy reason of space.[3] *The probability under consideration can then be brought into the form*

$$\kappa_{(\mu\nu;0)_t|11}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\sigma(\xi_1\eta_1,\ \xi_2\eta_2)+2A_tU(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)$$
$$+x_t\mathfrak{R}(\xi_1\eta_1,\ \xi_2\eta_2)+y_t\mathfrak{S}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)+z_t\mathfrak{T}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2).$$

Next, we observe the combination $(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)_{(\mu\nu;\,0)_t|1\nu}$ with $\mu_r,\ \nu_r>1$ for $1\leqq r\leqq t$ and $\nu\equiv\nu_{t+1}>1$. The defining equation of its reduced probability is given by

$$\kappa_{(\mu\nu;\,0)_t|1\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\sum\kappa_{(\mu\nu;\,0)_t|11}(\alpha\beta;\ \xi_1\eta_1,\ ab)\kappa_{\nu-1}(ab;\ \xi_2\eta_2).$$

Introducing three quantities[4] defined by

$$\mathfrak{R}^*(\xi_1\eta_1,\ \xi_2\eta_2)=2\sum\mathfrak{R}(\xi_1\eta_1,\ ab)Q(ab;\ \xi_2\eta_2),$$
$$\mathfrak{S}^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=2\sum\mathfrak{S}(\alpha\beta;\ \xi_1\eta_1,\ ab)Q(ab;\ \xi_2\eta_2),$$
$$\mathfrak{T}^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=2\sum\mathfrak{T}(\alpha\beta;\ \xi_1\eta_1,\ ab)Q(ab;\ \xi_2\eta_2),$$

we get the desired formula in the form

$$\kappa_{(\mu\nu;\,0)_t|1\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\sigma_{1\nu}(\xi_1\eta_1,\ \xi_2\eta_2)+A_t\{\bar{A}_{\xi_2\eta_2}Q(\alpha\beta;\ \xi_1\eta_1)$$
$$+2^{-\nu+1}V(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\}+x_t\{\tfrac{1}{2}\bar{A}_{\xi_2\eta_2}R(\xi_1\eta_1)+2^{-\nu+1}\mathfrak{R}^*(\xi_1\eta_1,\ \xi_2\eta_2)\}$$
$$+y_t\{\tfrac{1}{2}\bar{A}_{\xi_2\eta_2}S(\alpha\beta;\ \xi_1\eta_1)+2^{-\nu+1}\mathfrak{S}^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\}$$
$$+z_t\{\tfrac{1}{2}\bar{A}_{\xi_2\eta_2}T(\alpha\beta;\ \xi_1\eta_1)+2^{-\nu+1}\mathfrak{T}^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\}.$$

2. Mother-descendant combinations

We shall now consider a mother-descendant combination designated by $(\alpha\beta;\ \xi\eta)_{(\mu\nu;\,0)_t|n}$ with $\mu_r,\ \nu_r>1$ for $1\leqq r<t$ and with any $\mu_t,\ \nu_t\geqq 1$. Its reduced probability is given either by

$$\kappa_{(\mu\nu;\,0)_t|n}(\alpha\beta;\ \xi\eta)\equiv\kappa_{(\mu\nu;\,0)_{t-1}|\mu_t\nu_t;\,n}(\alpha\beta;\ \xi\eta)$$
$$=\sum\kappa_{(\mu\nu;\,0)_{t-1}|\mu_t\nu_t}(\alpha\beta;\ ab,\ cd)\varepsilon_n(ab,\ cd;\ \xi\eta)$$

3) Full tables for the values of these quantities will be listed in a paper which will shortly appear in Bull. Tokyo Inst. Tech. (1955) as the sequel of the papers referred to in 1).

4) For full tables for the values of these quantities, cf. also the paper cited in 3).

or equivalently by

$$\kappa_{(\mu\nu;0)_t|n}(\alpha\beta;\ \xi\eta)=\sum\kappa_{(\mu\nu;0)_t|\mu n}(\alpha\beta;\ ab,\ \xi\eta) \qquad \text{for any } \mu.$$

Here two systems are distinguished according to $n=1$ and $n>1$. For the lowest system $n=1$, *we can write down the desired formula in the form*

$$\kappa_{(\mu\nu;0)_t|1}(\alpha\beta;\ \xi\eta)=\bar{A}_{\xi\eta}+\Lambda_t Q(\alpha\beta;\ \xi\eta)+\tfrac{1}{2}x_t R(\xi\eta)$$
$$+\tfrac{1}{2}y_t S(\alpha\beta;\ \xi\eta)+z_t T(\alpha\beta;\ \xi\eta).$$

For generic system $n>1$, we may rather use, instead of the above-mentioned defining equations, an alternative equation

$$\kappa_{(\mu\nu;0)_t|n}(\alpha\beta;\ \xi\eta)=\sum\kappa_{(\mu\nu;0)_t|1}(\alpha\beta;\ ab)\kappa_{n-1}(ab;\ \xi\eta),$$

which readily yields the formula

$$\kappa_{(\mu\nu;0)_t|n}(\alpha\beta;\ \xi\eta)=\bar{A}_{\xi\eta}+\Lambda_t 2^{-n+1}Q(\alpha\beta;\ \xi\eta) \qquad \textit{for } n>1.$$

3. Descendants combinations

Elimination of mother's type in a mother-descendants combination leads to a corresponding descendants combination. In considering the probability thus defined by

$$\sigma_{(\mu\nu;0)_t|\mu\nu}(\xi_1\eta_1,\ \xi_2\eta_2)=\sum\bar{A}_{ab}\kappa_{(\mu\nu;0)_t|\mu\nu}(ab;\ \xi_1\eta_1,\ \xi_2\eta_2),$$

we distinguish three systems according to $\mu=\nu=1$, $\mu=1<\nu$ (or $\mu>1=\nu$), and $\mu,\ \nu>1$.

In case of the first and lowest system, *we obtain the formula*

$$\sigma_{(\mu\nu;0)_t|11}(\xi_1\eta_1,\ \xi_2\eta_2)=\sigma(\xi_1\eta_1,\ \xi_2\eta_2)+(x_t+2z_t)\Re(\xi_1\eta_1,\ \xi_2\eta_2),$$

while in case of the second system, *the formula is put into the form*

$$\sigma_{(\mu\nu;0)_t|1\nu}(\xi_1\eta_1,\ \xi_2\eta_2)=\sigma_{1\nu}(\xi_1\eta_1,\ \xi_2\eta_2)$$
$$+(x_t+2z_t)\{\tfrac{1}{2}\bar{A}_{\xi_2\eta_2}R(\xi_1\eta_1)+2^{-\nu+1}\Re^*(\xi_1\eta_1,\ \xi_2\eta_2)\}.$$

Finally, in case of the last and generic system, *the formula becomes*

$$\sigma_{(\mu\nu;0)_t|\mu\nu}(\xi_1\eta_1,\ \xi_2\eta_2)=\sigma_{\mu\nu}(\xi_1\eta_1,\ \xi_2\eta_2)+(x_t+2z_t)2^{-\lambda}\bar{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\ \xi_2\eta_2).$$

4. Distribution of genotypes in a generation of descendant

Elimination of mother's type in a mother-descendant combination or of a descendant's type in a descendants combination leads to a corresponding distribution of genotypes in the generation of an uneliminated descendant. We now consider the distribution defined by

$$\bar{A}_{(\mu\nu;0)_t|n}(\xi\eta)=\sum\bar{A}_{ab}\kappa_{(\mu\nu;0)_t|n}(ab;\ \xi\eta)=\sum\sigma_{(\mu\nu;0)_t|\mu n}(ab,\ \xi\eta) \qquad \text{for any } \mu.$$

It can be shown that *the final formulas are expressed by*

$$\bar{A}_{(\mu\nu;0)_t|1}(\xi\eta)=\bar{A}_{\xi\eta}+\tfrac{1}{2}(x_t+2z_t)R(\xi\eta),$$
$$\bar{A}_{(\mu\nu;0)_t|n}(\xi\eta)=\bar{A}_{\xi\eta} \qquad \text{for } n>1,$$

valid even when $\mu_t=\nu_t=1$ and when $\mu_t=1<\nu_t$ or $\mu_t>1=\nu_t$.

Thus, there appears a remarkable phenomenon analogous as stated in VI, § 4. In fact, *the distribution in a generation immediate* *after a consanguineous marriage deviates,* compared with the corresponding equilibrium distribution, *by the quantity $R(\xi\eta)$ multiplied by a positive factor $x_t/2+z_t$ which depends only on the generation-numbers constituting the assigned consanguinity.* The deviation vanishes out, however, soon in the next generation provided random matings take place.

5. Asymptotic behaviors of the probabilities

From explicit expressions derived for respective probabilities it can be readily deduced how they behave as anyone among the generation-numbers tends to infinity. For instance, we obtain the following limit equations:

$$\lim_{\mu\to\infty} \kappa_{(\mu\nu;0)_t|\mu\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\bar{A}_{\xi_1\eta_1}\kappa_{(\mu\nu;0)_t|\nu}(\alpha\beta;\ \xi_2\eta_2),$$

$$\lim_{\mu_w\to\infty} \kappa_{(\mu\nu;0)_t|\mu\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\kappa_{(\mu\nu;0)_{w-2}|\mu_{w-1}\nu_{w-1};\nu_w|(\mu'\nu';0)_{t-w}|\mu\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2),$$

$$\mu'_r\equiv\mu_{w+r}\quad\text{and}\quad\nu'_r\equiv\nu_{w+r}\quad\text{for}\quad 1\leqq r\leqq t-w.$$

Similar conclusions may be made also with respect to the probabilities such as $\kappa_{(\mu\nu;0)_t|n}(\alpha\beta;\ \xi\eta)$, $\sigma_{(\mu\nu;0)_t|\mu\nu}(\xi_1\eta_1,\ \xi_2\eta_2)$ and $\bar{A}_{(\mu\nu;0)_t|n}(\xi\eta)$.

Asymptotic behaviors as t tends to infinity can be deduced in a similar manner as in VI, §5 or VII, §5. For instance, we get the limit relations:

$$\bar{A}_{ii}+\underline{\zeta}R(ii)\leqq\varliminf_{t\to\infty}\kappa_{(\mu\nu;0)_t|1}(\alpha\beta;\ ii)\leqq\varlimsup_{t\to\infty}\kappa_{(\mu\nu;0)_t|1}(\alpha\beta;\ ii)\leqq\bar{A}_{ii}+\bar{\zeta}R(ii),$$

$$\bar{A}_{ij}+\bar{\zeta}R(ij)\leqq\varliminf_{t\to\infty}\kappa_{(\mu\nu;0)_t|1}(\alpha\beta;\ ij)\leqq\varlimsup_{t\to\infty}\kappa_{(\mu\nu;0)_t|1}(\alpha\beta;\ ij)\leqq\bar{A}_{ij}+\underline{\zeta}R(ij),$$

where we put

$$\bar{\lambda}=\varlimsup_{r\to\infty}\lambda_r\equiv\varlimsup_{r\to\infty}(\mu_r+\nu_r)-1,\qquad \underline{\lambda}=\varliminf_{r\to\infty}\lambda_r\equiv\varliminf_{r\to\infty}(\mu_r+\nu_r)-1,$$

$$\underline{\zeta}=1/(2^{\bar{\lambda}}-1),\qquad\qquad \bar{\zeta}=1/(2^{\underline{\lambda}}-1),$$

for which, in view of our assumption $\mu_r\geqq 2$ and $\nu_r\geqq 2$ for any r, there hold the estimating inequalities

$$3\leqq\underline{\lambda}\leqq\bar{\lambda}\leqq\infty,\qquad 0\leqq\underline{\zeta}\leqq\zeta\leqq 1/7.$$