35. *Prolongation of the Homeomorphic Mapping*

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1. Mr. G. Choquet enunciated the following theorem[1] "any homeomorphism between two closed bounded subsets of 2-dimensional Euclidean spaces contained in a 3-dimensional Euclidean space can be extended to a homeomorphism of 3-dimensional Euclidean space onto itself".

In this paper we shall give a solution of an analogous theorem in the case of two dimensions i.e. "*any homeomorphism between two closed bounded subsets of* 1-*dimensional Euclidean spaces contained in a* 2-*dimensional Euclidean space can be extended to a homeomorphism of* 2-*dimensional Euclidean space onto itself*".

Let E^i, L^i, F^i $(i=1, 2)$ and $\xi=f(x)$ be respectively the 2-dimensional Euclidean spaces (which are supposed hereafter to be two planes of complex numbers), the real axes of E^i, a closed bounded subsets of L^i, and a given homeomorphism between F^1 and F^2.

Now, we know that a homeomorphism between two Jordan arcs in a Euclidean plane can be extended to that of the whole plane by using the correspondences between the boundaries in conformal mappings[2] and the correspondence between the corresponding radii of two unit circles. Therefore, in order to prove the theorem, it is sufficient to show that we can construct a Jordan arc J which has the following properties: J is homeomorphic to the closed interval $[a, b]$, where a and b are the two end-points of F^1, and this homeomorphism between J and $[a, b]$ is an extension of the given homeomorphism $\xi=f(x)$.

2. In the first place, we assume that F^i is totally disconnected.

Since the derived set $(F^1)'$ is closed, $L^1-(F^1)'$ is open in L^1, therefore $L^1-(F^1)'$ is a sum set of an at most enumerable number of disjoint open intervals. As the number of those intervals, whose lengths are longer than a positive number $\rho>0$, is finite, we can enumerate all of the bounded disjoint open intervals in the order of their lengths. We denote them by

$$(x^*_{n,1}, x^*_{n,2}), \quad x^*_{n,1}(\in(F^1)') < x^*_{n,2}(\in(F^1)'), \quad x^*_{n,2}-x^*_{n,1} \geqq x^*_{n+1,2}-x^*_{n+1,1}$$

(where $x^*_{n,2} > x^*_{n+1,2}$ in the case of equality), $n=1, 2, 3, \dots$. Since $(x^*_{n,1}, x^*_{n,2}) \cap F^1$ can have only isolated points[3] of F^1, the order type

1) Comptes Rendus de l'Académie des Sciences de Paris, **219**, 542 (1944).

2) Cf. Hurwitz-Courant: Funktionentheorie, 400-405 (1929).

3) Inversely, it is clear that any isolated point F^1 belongs to one of the sets $(x^*_{n,1}, x^*_{n,2}) \cap F^1$, $n=1, 2, 3, \dots$

η of the points of that set must be equal to m_n (finite $\geqq 0$), ω, ω^*, or $\omega^*+\omega$. We shall denote all of the points of $[x^*_{n,1}, x^*_{n,2}] \cap F^1$ by

$$x_{n,1}=x^*_{n,1} < x_{n,2} < x_{n,3} < \cdots < x_{n,m_n+2}=x^*_{n,2},$$

$$x_{n,1}=x^*_{n,1} < x_{n,2} < x_{n,3} < \cdots < x_{n,m} < \cdots \to x^*_{n,2},$$

or $$x_{n,1}=x^*_{n,2} > x_{n,2} > x_{n,3} > \cdots > x_{n,m} > \cdots \to x^*_{n,1},$$

when the type η is finite (including 0), ω or ω^* respectively. In the case of $\omega^*+\omega$ type, the set $f([x^*_{n,1}, \frac{1}{2}(x^*_{n,1}+x^*_{n,2})] \cap F^1)$[4] is closed in L^2 and so it has two end-points, for $[x^*_{n,1}, \frac{1}{2}(x^*_{n,1}+x^*_{n,2})] \cap F^1$ is closed in L^1. Therefore at least one of these two end-points are different from $\xi^*_{n,1}=f(x^*_{n,1})$, and we shall denote it by $\xi^{(1)}_{n,1} (\neq \xi^*_{n,1})$. We write $x^{(1)}_{n,1}=f^{-1}(\xi^{(1)}_{n,1})$ and denote all of the points of $[x^*_{n,1}, x^*_{n,2}] \cap F^1$ by $x^*_{n,1} \leftarrow \cdots < x^{(1)}_{n,m} < \cdots < x^{(1)}_{n,2} < x^{(1)}_{n,1} < x^{(2)}_{n,1} < \cdots < x^{(2)}_{n,m} < \cdots \to x^*_{n,2}$.

Before we construct the Jordan arc J having the desired properties, we shall define some definitions of deformations of a family of arcs and give a lemma.

(1°) *Deformation* $\mathfrak{D}(\xi, \mathrm{I})$: Let ξ be an isolated point of F^2 and let $\gamma_\nu=\widehat{\xi_{2\nu-1}\,\xi_{2\nu}}$, $\nu=1, 2, \dots, m$, be a finite family of upper semi-circular arcs[5] which jump over the point ξ (i.e. $\xi_{2\nu-1}<\xi<\xi_{2\nu}$) and have following properties:

(i) $\xi_{2\nu-1}, \xi_{2\nu} \in L^2$, but ξ_λ does not necessarily belong to F^2,

(ii) $\xi_1 \leqq \xi_3 \leqq \cdots \leqq \xi_{2m-1} < \xi < \xi_{2m} \leqq \xi_{2m-2} \leqq \cdots \leqq \xi_2$,

(iii) any two of γ_ν do not cross each other i.e. the intersecting point must be one of their end-points,

(iv) and three of them have no common point.

Then, there exists a neighbourhood $V_{\rho(\xi)}(\xi)$ such as $V_{\rho(\xi)}(\xi) \cap F^2\{\xi\} = \{\xi\}$, where $\rho(\xi)$ is a positive number and $\{\xi\}$ denotes a set containing only one point ξ. We shall consider a tunnel $T(\xi)$ in the lower half plane, by drawing two semi-circular arcs from $\xi-\frac{k}{3}\rho(\xi)$ to $\xi+\frac{k}{3}\rho(\xi)$ in the lower half plane, $k=1, 2$. Draw m concentric semi-circular arcs $\widehat{\zeta_{2\nu-1}\,\zeta_{2\nu}}$ in the inside of the tunnel $T(\xi)$, where $\zeta_{2\nu-1}, \zeta_{2\nu} \in L^2-F^2$, $\nu=1, 2, \dots, m$, and

$$\xi-\frac{2}{3}\rho(\xi) < \zeta_{2m-1} < \zeta_{2m-3} < \cdots < \zeta_1 < \xi-\frac{1}{3}\rho(\xi),$$

$$\zeta_{2j-3}-\zeta_{2j-1}=\xi-\frac{1}{2}\rho(\xi)-\zeta_1=\zeta_{2m-1}-\left(\xi-\frac{2}{3}\rho(\xi)\right), \; j=1, 2, \dots, m,$$

$$\xi+\frac{1}{3}\rho(\xi) < \zeta_2 < \zeta_4 < \cdots < \zeta_{2m} < \xi+\frac{2}{3}\rho(\xi),$$

$$\zeta_{2j}-\zeta_{2(j-1)}=\zeta_2-\left(\xi+\frac{1}{3}\rho(\xi)\right)=\xi+\frac{2}{3}\rho(\xi)-\zeta_{2m}, \; j=1, 2, \dots, m.$$

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4) We considered this set in order to avoid Zermelo's axiom. As the point $\xi^{(1)}_{n,1}$, in general, we can adopt the end-point of $f([x^*_{n,1}, x^*_{n,2}] \cap (F^1 - V_\rho(x^*_{n,2})))$, where $V_\rho(x^*_{n,2})$ is a ρ-neighbourhood of the point $x^*_{n,2}$.

5) This means semi-circular arc drawn in the upper half plane.

Join the points ξ_λ and ζ_λ by the upper semi-circular arcs, $\lambda=1, 2, 3, \dots, 2m$. And we shall denote m arcs thus obtained by the same symbols γ_ν, $\nu=1, 2, \dots, m$, for simplification.

We shall denote this deformation of a family of arcs $\gamma_1, \gamma_2, \dots, \gamma_m$ by $\mathfrak{D}(\xi, \mathrm{I})$. Then,

(2,1) any deformed γ_ν has *the same length* and *the same end-points* of the old γ_ν. And the family of deformed $\gamma_1, \gamma_2, \dots, \gamma_m$ has also the above properties (iii) and (iv).

(2°) *Deformation* $\mathfrak{D}(\xi^*, \mathrm{II})$: Let ξ^* be a point of $(F^2)'$, and let $\gamma_\nu=\widehat{\xi_{2\nu-1}\xi_{2\nu}}$, $\nu=1, 2, \dots$, be a family of upper semi-circular arcs jumping over the point ξ^* (i.e. $\xi_{2\nu-1}<\xi^*<\xi_{2\nu}$) and having the properties (i), (iii), (iv), and

(ii′) $\xi_1 \leqq \xi_3 \leqq \cdots \leqq \xi_{2\nu-1} \leqq \cdots \to \xi^* \leftarrow \cdots \leqq \xi_{2\nu} \leqq \cdots \leqq \xi_4 \leqq \xi_2$.

By virtue of (iv), the equalities in (ii′) never succeed. Therefore, if $\xi_1=\xi_3$ then $\xi_3<\xi_5$, and so there exist two points $\zeta_{(1,1)}, \zeta_{(1,2)} \in L^2-F^2$ such as $\xi_3<\zeta_{(1,1)}<\zeta_{(1,2)}<\xi_5$,[6] and in this case it must be $\xi_2>\xi_4$ and so there exists a point $\zeta_{(2,1)} \in L^2-F^2$ such as $\xi_2>\zeta_{(2,1)}>\xi_4$.[7] If $\xi_1<\xi_3$, then there exists a point $\zeta_{(1,1)} \in L^2-F^2$ such as $\xi_1<\zeta_{(1,1)}<\xi_3$; and in this case there exists a point $\zeta_{(2,1)}$ such as $\xi_2>\zeta_{(2,1)}>\xi_4$, or two points $\zeta_{(2,1)}, \zeta_{(2,2)}$ such as $\xi_4>\zeta_{(2,1)}>\zeta_{(2,2)}>\xi_6$ provided $\xi_2>\xi_4$ or $\xi_2=\xi_4>\xi_6$ respectively, and so on. Thus we have two sequences of points as follows:

$$\zeta_{(1,1)}<\zeta_{(1,2)}<\cdots<\zeta_{(1,\nu)}<\cdots\to\xi^*;\ \zeta_{(2,1)}>\zeta_{(2,2)}>\cdots>\zeta_{(2,\nu)}>\cdots\to\xi^*,$$

where the numbers of $\zeta_{(1,\lambda)}$ such as $\zeta_{(1,\lambda)}<\xi_{2\nu+1}$ or $\zeta_{(2,\nu)}>\xi_{2\nu+2}$ is equal to ν, for any index ν such as $\xi_{2\nu-1}<\xi_{2\nu+1}$ or $\xi_{2\nu+2}<\xi_{2\nu}$ respectively. Since L^2-F^2 is open in L^2, there exist disjoint neighbourhoods $V_{\rho(k,\nu)}(\zeta_{(k,\nu)}) \subset L^2-F^2$, $\rho(k, \nu)>0$; $k=1, 2$; $\nu=1, 2, 3, \dots$. Put into $\rho(\nu)=\min\left(\frac{1}{2}\rho(1, \nu), \frac{1}{2}\rho(2, \nu)\right)$, and consider the tunnels $T(\xi^*, \nu)$, $\nu=1, 2, \dots$, by drawing semi-circular arcs in the lower half plane from $\zeta_{(1,\nu)}-\rho(\nu)$, $\zeta_{(1,\nu)}+\rho(\nu)$ to $\zeta_{(2,\nu)}+\rho(\nu)$, $\zeta_{(2,\nu)}-\rho(\nu)$ respectively. Deform the arc γ_ν with respect to the tunnel $T(\xi^*, \nu)$ having the same index, similarily in the case of (1°). And we shall denote the deformed arcs by the same symbols γ_ν for simplification, $\nu=1, 2, \dots$.

We shall denote this deformation of a family of arcs $\gamma_1, \gamma_2, \dots$ by $\mathfrak{D}(\xi^*, \mathrm{II})$. Then,

6) In order to avoid Zermelo's axiom, we consider the most right open interval $(\xi_{(3)}, \xi_{(5)})$ among the longest open intervals (it is clear that the number of them is finite), which are the components of $[\xi_3, \xi_5]-F^2$, and put into $\zeta_{(1,k)}=\xi_{(3)}+\frac{k}{3}(\xi_{(5)}-\xi_{(3)})$, $k=1, 2$. Those that follow are the same.

7) We consider the most right open interval $(\xi_{(4)}, \xi_{(2)})$ among the longest open intervals $\subseteqq [\xi_4, \xi_2]-F^2$, and put into $\zeta_{(2,1)}=\frac{1}{2}(\xi_{(4)}+\xi_{(2)})$. Those that follow are the same.

(2,2) any deformed γ_ν has *the same length* and *the same endpoints* of the old γ_ν. And the family of deformed $\gamma_1, \gamma_2, \dots$ has also the above properties (iii) and (iv).

Lemma. *Let (α, β) any open interval, where $\alpha < \beta$, the numbers of pairs of points ξ, $\hat{\xi}$ of F^2 such as $\xi < \alpha < \beta < \hat{\xi}$ or $\hat{\xi} < \alpha < \beta < \xi$, where $\hat{x} = f^{-1}(\hat{\xi})$ is the next right point of $x = f^{-1}(\xi)$ in F^1, is finite.*

(Proof). Assume that there exists an infinite number of pairs ξ, $\hat{\xi}$ having above properties, then there exists at least an accumulated point x^* of the set of the above points x_s or that of the set of the above points $\hat{x}_s$, for F^1 is bounded. And so, in the first case for example, there exists a sequence $x^{(1)}, x^{(2)}, \dots, x^{(n)}, \dots \to x^*$. Since the open intervals $(x^{(n)}, \hat{x}^{(n)})$, $n = 1, 2, \dots$, are disjoint, the lengths of the intervals $(x^{(n)}, \hat{x}^{(n)})$ tend to 0 when $n \to \infty$. Therefore the lengths $|\hat{\xi}^{(n)} - \xi^{(n)}|$ of the intervals $(\xi^{(n)}, \hat{\xi}^{(n)})$ tend to 0 when $n \to \infty$, where $\xi^{(n)} = f(x^{(n)})$, $\hat{\xi}^{(n)} = f(\hat{x}^{(n)})$. This result contradicts the assumption of the lemma: $|\hat{\xi}^{(n)} - \xi^{(n)}| > \beta - \alpha > 0$ for all $n = 1, 2, \dots$.

3. (1°) In the first place, we shall consider the points of $f([x^*_{1,1}, x^*_{1,2}] \cap F^1)$.

a) The case when the order type η of the points of $(x^*_{1,1}, x^*_{1,2}) \cap F^1$ is $m_1 (\geqq 0)$, ω, or ω^*. We draw the upper semi-circular arc $\gamma_{1,1} = \widehat{\xi_{1,1}\,\xi_{1,2}}$. If the arc $\gamma_{1,1}$ jumps over[8] the point $\xi_{1,3}$, then we practice the deformation $\mathfrak{D}(\xi_{1,3}, \mathrm{I})$ of the arc $\gamma_{1,1}$ already drawn, and next we draw the upper semi-circular arc $\gamma_{1,2} = \widehat{\xi_{1,2}\,\xi_{1,3}}$. If the deformed $\gamma_{1,1}$ or $\gamma_{1,2}$ has the upper semi-circular part jumping over the point $\xi_{1,4}$, then we practice the deformation $\mathfrak{D}(\xi_{1,4}, \mathrm{I})$ of a family of the above parts of $\gamma_{1,1}$ and $\gamma_{1,2}$; next we draw the upper semi-circular arc $\gamma_{1,3} = \widehat{\xi_{1,3}\,\xi_{1,4}}$. Then, by virtue of (2,1), we can practice the deformation $\mathfrak{D}(\xi_{1,5}, \mathrm{I})$ of a family of the upper semi-circular parts (jumping over the point $\xi_{1,5}$) of the arcs already drawn, and draw the upper semi-circular arcs $\gamma_{1,4} = \widehat{\xi_{1,4}\,\xi_{1,5}}$, and so on. If $\eta = m_1 > 0$, then we consider a family of the upper semi-circular parts (jumping over the point $\xi^*_{1,2}$) of the arcs $\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{1,m_1}$, and practice $\mathfrak{D}(\xi^*_{1,2}, \mathrm{II})$ of this family of arcs, and we draw the upper semi-circular arc $\gamma_{1,m_1+1} = \widehat{\xi_{1,m_1+1}\,\xi^*_{1,2}}$.

Then, by virtue of (2,1) and (2,2), we obtain the Jordan arc J_1 which is homeomorphic to $I_1 = [x^*_{1,1}, x^*_{1,2}]$, $[x^*_{1,1}, x^*_{1,2}]$, or $[x^*_{1,1}, x^*_{1,2}]$ when η is equal to a finite number (including 0), ω or ω^* respectively, and this homeomorphism between J_1 and I_1 is an extension of $\xi = f(x)$

8) Cf. n° 2.

where $x \in I_1 \cap F^1$.

b) The case when the order type η is equal to $\omega^* + \omega$. At the first place, we consider the points $\xi_{1,1}^{(1)}, \xi_{1,2}^{(1)}, \dots$ and construct successively the arcs $\gamma_{1,1}^{(1)}, \gamma_{1,2}^{(1)}, \dots, \gamma_{1,m}^{(1)}, \dots$ similarily at a). By virtue of (2,1) and the definition of $\xi_{1,1}^{(1)}$, any deformed $\gamma_{1,m}^{(1)}$ does not jump over this point $\xi_{1,1}^{(1)}$, and so there is no need of practicing the deformation $\mathfrak{D}(\xi_{1,1}^{(1)}, \mathrm{I})$. Next we consider the point $\xi_{1,1}^{(2)}$ and a family of all of the upper semi-circular parts (jumping over the point $\xi_{1,1}^{(2)}$) of the arcs already drawn. Then, by virtue of the lemma and (2,1), this family is a finite family satisfing (i), (ii), (iii), and (iv), therefore we can practice the deformation $\mathfrak{D}(\xi_{1,1}^{(2)}, \mathrm{I})$ of this family, and we draw the upper semi-circular arc $\gamma_{1,1}^{(2)} = \widehat{\xi_{1,1}^{(1)} \xi_{1,1}^{(2)}}$. Next, we practice the deformation $\mathfrak{D}(\xi_{1,2}^{(2)}, \mathrm{I})$ and draw $\gamma_{1,2}^{(2)} = \widehat{\xi_{1,1}^{(2)} \xi_{1,2}^{(2)}}$ and so on. Then the arc J_1 thus obtained is homeomorphic to $(x_{1,1}^*, x_{1,2}^*)$ and this homeomorphism is an extension of $\xi = f(x)$, $x \in (x_{1,1}^*, x_{1,2}^*) \cap F^1$.

(2°) Next, we consider the point of $f([x_{2,1}^*, x_{2,2}^*] \cap F^1)$. A family of all of the upper semi-circular parts (jumping over the point $\xi_{2,1} (= \xi_{2,1}^*, \xi_{2,2}^*)$ or $\xi_{2,1}^{(1)}$) of the arcs already drawn, is an at most enumerable family. And so, by virtue of (2,1), (2,2), and *the lemma*, it is easy to verify that this family satisfies the properties (i), (ii′), (iii), (iv), or (i), (ii), (iii), (iv); therefore we can practice the deformation $\mathfrak{D}(\xi_{2,1}, \mathrm{II})$, or $\mathfrak{D}(\xi_{2,1}^{(1)}, \mathrm{I})$ of the above family of arcs, and so we can construct the Jordan arc J_2 similarily as (1°), and so on. Thus we obtain deformed $J_1, J_2, \dots, J_m, \dots$. As any tunnel has only a finite number of arcs inside of it, any accumulated point of $\bigcup_{1=n<\infty} J_n$ is that of $\bigcup_{1=n<\infty} J_n \cap F^2$. And the set $\varPhi$ consisted of all isolated points and $x_{n,1}^*, x_{n,2}^*$, $n=1, 2, \dots$, which are not the accumulated points of the isolated points of F^1, is dense in F^1 and $f(\varPhi)$ is dense in F^2. Therefore, if we make the point $\xi = \lim_{\lambda\to\infty} f(x(\lambda))$ correspond to $x = \lim_{\lambda\to\infty} x(\lambda)$ where $x(\lambda) \in \varPhi$, then it is clear that the closure $\overline{(\bigcup_{1=n<\infty} J_n)}$ is a Jordan arc J which is homeomorphic to the segment $[a, b]$ and this homeomorphism is an extension of $\xi = f(x)$, $x \in F^1$.

4. At the end, we consider the case such as F^i is not totally disconnected. If a component K of F^1 is a closed interval, then $f(K)$ is a component of F^2 and so it is a closed interval. And the end-points of the components play the role of the points of F^1 in the case of n° 2 and n° 3. The sum set of the arcs constructed similarily as n° 3 and the components of F^2 constructs a Jordan arc J having all of the desired properties.

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