49. Integrability of Trigonometrical Series. II

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1. We shall consider the trigonometrical series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{1}$$

Given a sequence $c_0, c_1, c_{-1}, \ldots$ such that $c_n \to 0$, let $c_0^* \geqq c_1^* \geqq c_{-1}^* \geqq c_2^* \geqq \cdots$ be the sequence $|c_0|, |c_1|, |c_{-1}|, \ldots$ arranged in the descending order of magnitude.

Recently R. P. Boas [1] proved the following

Theorem B. *If $1<q\leqq 2$, $1\leqq p<q/(q-1)$, and $\alpha<1-q/p'$, then (1) is the Fourier series of a function of L^p if $c_n\to 0$ and*

$$\sum_{n=-\infty}^{\infty} |c_{n+m}-c_{n-m}|^q = O(m^\alpha) \tag{2}$$

as $m\to\infty$ through the multiples of some fixed integer.

If $\alpha\geqq 1-q/p'$ the conclusion no longer holds.

In this paper we prove the following theorems.

Theorem 1. *If $q\geqq 2$, $p\geqq 1$, and $0<\alpha<q/p-1$, then (1) is the Fourier series of a function of L^p if $c_n\to 0$ and*

$$\sum_{n=-\infty}^{\infty} (c_{n+m}-c_{n-m})^{*q} n^{q-2} = O(m^\alpha) \tag{3}$$

as $m\to\infty$ through the multiples of some fixed integer.

If $\alpha=q/p-1$, $\alpha>q-2$, the conclusion no longer holds.

Theorem 2. *If $q\geqq 2$, $p\geqq 1$, $q'\leqq r\leqq q$, $\mu=1/r+1/q-1$, and $0<\alpha<q/p-1$, then (1) is the Fourier series of a function of L^p if $c_n\to 0$ and*

$$\sum_{n=-\infty}^{\infty} |c_{n+m}-c_{n-m}|^r (|n|+1)^{-\mu r} = O(m^{\alpha r/q}) \tag{4}$$

as $m\to\infty$ through the multiples of some fixed integer.

If $\alpha\geqq q/p-1$ the conclusion no longer holds.

In Theorem 2, if $r=q'$ then it becomes Theorem B, and if $r=q$ then it becomes Theorem 1 except star. Hence Theorem 2 contains Theorem B formally but Theorems 1 and 2 are mutually exclusive.

The proofs of Theorems 1 and 2 are similar to that of Theorem B, the difference being to use the following Theorems HL 1 and HL 2 [2], respectively, instead of the Hausdorff-Young theorem. We prove here Theorem 1 only.

Theorem HL 1. *If $q\geqq 2$ then (1) is the Fourier series of a function $f(x)$ of L^q and*

$$\left(\int_{-\pi}^{\pi}|f(x)|^q dx\right)^{1/q} \leqq A_q\left\{\sum_{n=-\infty}^{\infty} c_n^{*q} n^{q-2}\right\}^{1/q}$$

where A_q depends on q only, if $c_n \to 0$ and

$$\left\{\sum_{n=-\infty}^{\infty} c_n^{*q} n^{q-2}\right\}^{1/q} < \infty.$$

Theorem HL 2. *If $q \geqq 2$, $q' \leqq r \leqq q$, and $\mu = 1/r + 1/q - 1$, then (1) is the Fourier series of a function $f(x)$ of L^q and*

$$\left(\int_{-\pi}^{\pi}|f(x)|^q dx\right)^{1/q} \leqq A_q\left\{\sum_{n=-\infty}^{\infty} |c_n|^r(|n|+1)^{-\mu r}\right\}^{1/r}$$

where A_q depends on q only, if $c_n \to 0$ and

$$\left\{\sum_{n=-\infty}^{\infty} |c_n|^r(|n|+1)^{-\mu r}\right\}^{1/r} < \infty.$$

2. Proof of the first part of Theorem 1. If (3) is satisfied for $m = k$, then by Theorem HL 1 $c_{n+k} - c_{n-k}$ are the nth Fourier coefficients of a function $\varphi_k(t)$ of L^q, i.e.,

$$c_{n+k} - c_{n-k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\varphi_k(t)dt.$$

The function $\varphi(t) = \varphi_k(t)/\sin kt$ belongs to L^q except perhaps in neighbourhoods of the points 0, $\pm\pi/k$, $\pm 2\pi/k, \dots, \pm\pi$. We have to show that $\varphi(t)$ actually belongs to L^p and has (c_n) as its Fourier coefficients. Now if m is a multiple of k, $\varphi(t)\sin mt$ is integrable, and

$$\int_{-\pi}^{\pi} e^{-int}\varphi(t)\sin mt\, dt = \int_{-\pi}^{\pi} e^{-int}\varphi_k(t)\frac{\sin mt}{\sin kt}dt$$
$$= 2\pi(c_{n+m} - c_{n-m}).$$

Thus again by Theorem HL 1 we have

$$\int_{-\pi}^{\pi} |\varphi(t)\sin mt|^q\, dt \leqq A_q \sum_{n=-\infty}^{\infty}(c_{n+m} - c_{n-m})^{*q} n^{q-2},$$

where A_q depends on q only.

We shall prove the integrability of $\varphi(t)$ in a neighbourhood of $t=0$. By (3) and the inequality $\sin t \geqq 2t/\pi$ for $0 \leqq t \leqq \pi/2$

$$\int_0^{1/m} |\varphi(t)|^q t^q\, dt \leqq \frac{C}{m^q}\int_{-\pi}^{\pi}|\varphi(t)\sin mt|^q\, dt$$
$$\leqq \frac{C}{m^q}\sum_{n=-\infty}^{\infty}(c_{n+m} - c_{n-m})^{*q} n^{q-2} \leqq C m^{\alpha - q},$$

where C is an absolute constant. Since $|\varphi(t)\sin kt|^q$ is integrable, so is $t^p|\varphi(t)|^p$, $1 \leqq p \leqq q$. We put

$$F(t) = \int_0^t x^p|\varphi(x)|^p\, dx,$$

then for $\varepsilon < \pi/k$

$$\int_{1/m}^{\varepsilon} |\varphi(t)|^p\, dt = \int_{1/m}^{\varepsilon} x^{-p}\, dF(x)$$
$$= F(\varepsilon)\varepsilon^{-p} - F(1/m)m^p - p\int_{1/m}^{\varepsilon} F(x)x^{-p-1}dx.$$

By Hölder's inequality with index q/p we have

$$F(1/m)=\int_0^{1/m} x^p\,|\varphi(x)|^p\,dx \leqq \left\{\int_0^{1/m} x^q\,|\varphi(x)|^q\,dx\right\}^{p/q} m^{-(q-p)/q}$$

$$\leqq Cm^{\{(\alpha-p)p-(q-p)\}/q}=o(m^{-p}),$$

since $\alpha p<q-p$. Accordingly $F(1/m)m^p=o(1)$. And further $F(t)$ is a non-decreasing function of t, if $m=rk$ (r an integer) and $1/(r+1)k \leqq t\leqq 1/rk$, we have $F(t)\leqq Ct^u (u>p)$. Thus $F(x)x^{-p-1}$ is dominated by the integrable function x^{u-p-1} in a neighbourhood of 0, i.e.,

$$F(x)x^{-p-1}\leqq Cx^{u-p-1}.$$

Thus we see that $\varphi(x)$ is L^p integrable in a neighbourhood of 0. The same proof applies for the L^p integrability of $\varphi(x)$ in neighbourhoods of $\pm\pi/k, \pm 2\pi/k,\ldots, \pm\pi$.

3. **Proof of the second part of Theorem 1.** We shall consider the series used by R. P. Boas [1]

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

such that $c_n=n^{-\gamma}(0<\gamma<1/q)$ for $n>0$, $c_n=c_{-n}$, and $c_0=1$. Then $f(x)$ is of order $x^{\gamma-1}$ as $x\to 0$ and consequently belongs to L^p for $p<1/(1-\gamma)$ and not for $p\geqq 1/(1-\gamma)$.

We shall now estimate the order of the series

$$\sum_{n=0}^{\infty}(c_{n+m}-c_{n-m})^{*q}n^{q-2} \qquad \text{as } m\to\infty.$$

Writing $d_n=|c_{n+m}-c_{n-m}|$, we can see that $d_{m+1}\geqq d_m\geqq d_{m-1}\geqq d_{m-2}$, and more generally $d_{m+k}\geqq d_{m-k}\geqq d_{m+k+1}$ for $k\geqq\mu=(am)^{\delta}$, where $\delta=\gamma/(\gamma-1)$ and $a=\gamma^{1/\gamma}$. Thus we have

$$\sum_{n=1}^{\infty} d_n^{*q}n^{q-2}\leqq (d_{m+1}^q+2^{q-2}d_m^q+3^{q-2}d_{m-1}^q+\cdots+(2\mu)^{q-2}d_{m+\mu}^q)$$

$$+\sum_{n=1}^{m-\mu} d_n^q(2\mu+n)^{q-2}+\sum_{n=m+\mu+1}^{\infty} d_n^q n^{q-2}$$

$$\equiv S_1+S_2+S_3$$

say. Then, if $\gamma q<1$,

$$S_1=\left\{\left(1-\frac{1}{(2m+1)^{\gamma}}\right)^q+2^{q-2}\left(1-\frac{1}{(2m)^{\gamma}}\right)^q+3^{q-2}\left(1-\frac{1}{(2m-1)^{\gamma}}\right)^q\right\}$$

$$+\sum_{k=2}^{\mu}\left\{(2k)^{q-2}\left(\frac{1}{k^{\gamma}}-\frac{1}{(2m+k)^{\gamma}}\right)^q+(2k+1)^{q-2}\left(\frac{1}{k^{\gamma}}-\frac{1}{(2m-k)^{\gamma}}\right)^q\right\}$$

$$\leqq C\sum_{k=2}^{\mu}\frac{1}{k^{\gamma q-q+2}}\leqq C\mu^{-(\gamma q-q+1)}=Cm^{(-\gamma q+q-1)\gamma/(1-\gamma)},$$

$$S_2=\sum_{n=1}^{m-\mu}\left(\frac{1}{|n-m|^{\gamma}}-\frac{1}{(n+m)^{\gamma}}\right)(2\mu-n)^{q-2}$$

$$\leqq Cm^q\sum_{n=1}^{m-\mu}\frac{n^q}{(n+m)^{2q}(m-n)^{\gamma q}}(2\mu+n)^{q-2}$$

$$=Cm^q\left(\sum_{n=1}^{m/2}+\sum_{n=m/2+1}^{m-\mu}\right)\frac{n^q}{(n+m)^{2q}(m-n)^{\gamma q}}(2\mu+n)^{q-2}$$

$$\leqq Cm^{-\gamma q+q-2}\sum_{n=1}^{m/2}1+Cm^{q-2}\sum_{\mu<k\leqq m/2}k^{-\gamma q}=Cm^{q-\gamma q-1}$$

and

$$
\begin{aligned}
S_3 &= \sum_{n=m+\mu+1}^{\infty}\left(\frac{1}{|n-m|^{\gamma}}-\frac{1}{(n+m)^{\gamma}}\right)^{q} n^{q-2} \\
&\leqq Cm^{q}\sum_{n=m+\mu+1}^{\infty}\frac{n^{2q-2}}{(n+m)^{2q}(n-m)^{\gamma q}} \\
&= Cm^{q}\left(\sum_{n=m+\mu+1}^{2m}+\sum_{n=2m+1}^{\infty}\right)\frac{n^{2q-2}}{(n+m)^{2q}(n-m)^{\gamma q}} \\
&\leqq Cm^{q-2}\sum_{n=m+\mu+1}^{2m}\frac{1}{(n-m)^{\gamma q}}+Cm^{q}\sum_{n=2m+1}^{\infty}\frac{1}{(n-m)^{\gamma q+2}} \\
&= Cm^{q-\gamma q-1}.
\end{aligned}
$$

Collecting above estimations, we obtain

$$\sum_{n=1}^{\infty} d_n^{*q} n^{q-2} = O(m^{q-\gamma q-1}),$$

and hence, if we take γ such as $\alpha=q-\gamma q-1$, that is, $\gamma=1-(\alpha+1)/q$, then the condition (3) is satisfied, but since $q\gamma=q-\alpha-1$, we get $q\gamma<1$ when $q-2<\alpha$.

References

[1] R. P. Boas: Integrability of trigonometrical series II, Math. Zeits., **55**, 183–186 (1952).

[2] G. H. Hardy and J. E. Littlewood: Some new properties of Fourier constants, Math. Ann., **97**, 159-209 (1926); Jour. London Math. Soc., **6**, 3–9 (1931).

[3] A. Zygmund: Trigonometrical series, Warszawa (1935).