63. On the Property of Lebesgue in Uniform Spaces. II

By Kiyoshi ISÉKI
Kobe University

(Comm. by K. KUNUGI, M.J.A., May 13, 1955)

In this Note, we shall discuss the relation between Lebesgue property and uniformly continuity in a uniform space.*) The theorems to be proved are generalisations of some results by A. A. Monteiro and M. M. Peixoto (3).

Theorem 1. If a uniform space E is normal and every bounded continuous function is uniformly continuous, then any finite covering of E has the Lebesgue property.

Proof. Let F_1, F_2 be two closed sets such that $F_1 \frown F_2 = 0$. By a theorem of Urysohn, we can find a continuous function $f(x)$ on the uniform space E such that

(1) $$0 \leq f(x) \leq 1 \quad \text{on } E,$$

(2) $$f(x) = 0 \quad \text{for } x \in F_1,$$

and

(3) $$f(x) = 1 \quad \text{for } x \in F_2.$$

Since the function $f(x)$ is uniform continuous, for a given positive number ε less than 1, there is a surrounding V such that $V(a) \ni x, y$ implies

(4) $$|f(x) - f(y)| < \varepsilon.$$

Suppose that $V(F_1) \frown F_2 \neq 0$, then, for $x \in V(F_1) \frown F_2$, $y \in F_2$, $(x, y) \in V$, and $x \in F_2$, and hence $|f(x) - f(y)| < \varepsilon$ by (4). From (2) and (3) $|f(x) - f(y)| = 1$, which is a contradiction. Therefore any binary covering of E has the property of Lebesgue, and since E is normal, each finite covering of E has the Lebesgue property. Q.E.D.

Conversely, we shall prove the following

Theorem 2. If any covering of a uniform space E has the Lebesgue property, then any continuous function on E is uniformly continuous.

Proof. Let $f(x)$ be a continuous function on E. To prove that $f(x)$ is uniformly continuous, let $O_\alpha = f^{-1}(I_\alpha)$, where I_α is any open interval with the length ε. $\{O_\alpha\}$ is an open covering of E. Since E has the Lebesgue property, there is a surrounding V such that $V(a) \subset O_\alpha$ for some index α depending on a. Hence $V(a) \ni x, y$ implies

$$|f(x) - f(y)| \leq |f(x) - f(a)| + |f(a) - f(y)| < 2\varepsilon.$$

This shows that $f(x)$ is uniformly continuous.

---

*) For the definitions and properties of Lebesgue property in a uniform space, see K. Iséki (2). For the definition of uniformly continuity, see N. Bourbaki (1) or G. Nöbeling (4).

Remark. As easily shown, the hypothesis of Theorem 2 is replaced by the condition: *any countable covering of E has the Lebesgue property.*

References

1) N. Bourbaki: Topologie générale, Chap. 1-10 Hermann, Paris (1940-1949).
2) K. Iséki: On the property of Lebesgue in uniform spaces, Proc. Japan Acad., **31**, 220-221 (1955).
3) A. A. Monteiro and M. M. Peixoto: Le nombre de Lebesgue et la continuité uniformé, Port. Math., **10**, 105-113 (1951).
4) G. Nöbeling: Grundlagen der analytischen Topologie, Springer, Berlin (1954).