75. *Some Trigonometrical Series. XIV*

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1. E. Hille and G. Klein [1] have proved the following theorem:

Theorem 1. *If $f(t)$ is integrable and is not constant, then*

$$m_1(h) \leqq A\omega_1(h), \tag{1}$$

where

$$m_1(h) = \max_{0\leqq x\leqq 2\pi} \int_0^h |f(x+t)|\,dt,$$

$$\omega_1(h) = \max_{|t|\leqq h} \int_0^{2\pi} |f(x+t)-f(x)|\,dx.$$

Their proof depends on the theory of interpolation polynomials. We shall give here a direct proof of Theorem 1. Further we can prove its pth power analogue. That is,

Theorem 2. *If $f(t)$ belongs to the L^p-class and is not constant, then*

$$m_p(h) \leqq A\omega_p(h)/h^{p-1} \tag{2}$$

where

$$m_p(h) = \max_{0\leqq x\leqq 2\pi} \int_0^h |f(x+t)|^p\,dt,$$

$$\omega_p(h) = \max_{|t|\leqq h} \int_0^{2\pi} |f(x+t)-f(x)|^p\,dx.$$

Proof is similar to that of Theorem 1.[1)]

2. **Proof of Theorem 1.** Let $f(r, x)$ be the Poisson integral of $f(x)$, that is,

$$f(r, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)P_r(t)\,dt \tag{3}$$

where

$$P_r(t) = (1-r^2)/2\Delta_r(t),$$

$$\Delta_r(t) = 1-2r\cos t + r^2 = (1-r)^2 - 4r\sin^2 t/2.$$

It is well known [2] that $P_r(t)$ is non-negative and its integral in $(-\pi, \pi)$ equals to π. Then (3) gives

$$\begin{aligned} f'(r, x) &= -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)P'_r(t)\,dt \\ &= -\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x+t)-f(x)]P'_r(t)\,dt \end{aligned} \tag{4}$$

1) For a non-constant linear function, the equality in (2) holds, hence $1/h^{p-1}$ is the best possible factor.

where the dashes denote the differentiation with respect to x and t, respectively.

We have

$$f(t)=f(r, t)+[f(t)-f(r, t)],$$

$$\int_x^{x+h}|f(t)|\,dt \leqq \int_x^{x+h}|f(r, t)|\,dt+\int_x^{x+h}|f(t)-f(r, t)|\,dt=I+J.$$

We can suppose $x=0$ and let $r=1-h$. Let ξ be a Lebesgue point of $f(t)$, then $|f(r, \xi)|$ is less than a constant for all $r<1$. By (4),

$$I \leqq \int_0^h dt\left|\int_\xi^t f'(r, u)\,du\right|+Ah=K+Ah,$$

$$K \leqq A\int_0^h dt\int_\xi^t du\int_0^{2\pi}\frac{(1-r)v}{\Delta_r^2(v)}|f(u+v)-f(u)|\,dv,$$

since $P_r'(t)=-(1-r^2)\sin t/r\Delta_r^2(t)$. Hence

$$K \leqq Ah\int_0^h dt\sum_{k=0}^{[2\pi/h]+1}\int_{kh}^{(k+1)h}\frac{vdv}{(h^2+v^2)^2}\int_\xi^t|f(u+v)-f(u)|\,dv$$

$$\leqq A\omega_1(h)h\sum_{k=0}^{[2\pi/h]+1}\int_{kh}^{(k+1)h}\frac{v^2dv}{(h^2+v^2)^2}$$

$$\leqq A\omega_1(h)h\int_0^{2\pi}\frac{v^2dv}{(h^2+v^2)^2} \leqq A\omega_1(h).$$

On the other hand,

$$J=\frac{1}{\pi}\int_0^h dt\left|\int_0^{2\pi}[f(t+u)-f(t)]P_r(u)\,du\right|$$

$$\leqq \frac{1}{\pi}\int_0^{2\pi}P_r(u)\,du\int_0^h|f(t+u)-f(t)|\,dt$$

$$\leqq A\sum_{k=0}^{[2\pi/h]+1}\int_{kh}^{(k+1)h}P_r(u)\,du\int_0^h|f(t+u)-f(t)|\,dt$$

$$\leqq A\sum_{k=0}^{[2\pi/h]+1}\int_{kh}^{(k+1)h}P_r(u)\,du\left(\int_0^h|f(t+u)-f(t+kh)|\,dt+\int_0^{kh}|f(t+h)-f(t)|\,dt\right)$$

$$\leqq A\omega_1(h).$$

Thus Theorem 1 is proved.

We can also prove the theorem using the Cesàro mean $\sigma_n(x)$ $(h=1/n)$ instead of the Poisson mean $f(\nu, t)$ $(\nu=1-h)$.

3. **Proof of Theorem 2.** Let $f(t)$ be a function of the L^p-class. Using the notation in §2, we write

$$\left(\int_x^{x+h}|f(t)|^p\,dt\right)^{1/p} \leqq \left(\int_x^{x+h}|f(r, h)|^p\,dt\right)^{1/p}+\left(\int_x^{x+h}|f(t)-f(r, t)|^p\,dt\right)^{1/p}$$

$$=I_p+J_p.$$

Then, taking $x=0$,

$$I_p \leqq \left(\int_0^h dt \left| \int_\xi^t f'(r, u)\, du \right|^p \right)^{1/p} + Ah^{1/p} = I_{p,1} + Ah^{1/p},$$

$$I_{p,1}^p \leqq A \int_0^h dt \left(\int_\xi^t du \int_0^{2\pi} \frac{hv}{\Delta_r^2(v)} \,|\, f(u+v) - f(u) \,|\, dv \right)^p$$

$$\leqq A \int_0^h dt \left(\int_0^{2\pi} \frac{v^2 dv}{\Delta_r^2(v)} \int_\xi^t |\, f(u+h) - f(u) \,|\, du \right.$$

$$\left. + h \int_0^{2\pi} \frac{v dv}{\Delta_r^2(v)} \int_\xi^t \left| f(u+v) - f\left(u + \left[\frac{v}{h}\right] h\right) \right| du \right)^p$$

$$\leqq A(\omega_1(h))^p / h^{p-1} \leqq A\omega_p(h)/h^{p-1}.$$

Finally

$$J_p = \int_0^h dt \left| \int_0^{2\pi} [\, f(u+t) - f(t) \,] P_r(u)\, du \right|^p$$

$$\leqq \int_0^h dt \int_0^{2\pi} P_r(u) \,|\, f(u+t) - f(t) \,|^p\, du$$

$$\leqq A\omega_p(h).$$

Thus Theorem 2 is proved.

References

[1] E. Hille and G. Klein: Duke Math. Journ., **21** (1954).
[2] A. Zygmund: Trigonometrical series, Warszawa (1935).