100. *On a Homogeneous Space with Invariant Affine Connection*

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1. Preliminaries. Let G/H be a reductive homogeneous space,[1] where G is a connected Lie group and H a closed subgroup of G. Then in the Lie algebra $\mathfrak{g}$ of G there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$ (direct) and $\mathrm{ad}(H)\cdot\mathfrak{m}\subset\mathfrak{m}$, where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to H. If we denote by $[\mathfrak{h}, \mathfrak{m}]$ the subspace spanned by all elements of the form $[U, X]$, $U\in\mathfrak{h}$, $X\in\mathfrak{m}$, we have then $[\mathfrak{h}, \mathfrak{m}]\subset\mathfrak{m}$. $\mathfrak{m}$ may be identified with the tangent space at the point $p_0=H$ of G/H. Throughout this note we assume that G is almost effective on G/H as a transformation group, that is to say that H does not contain any positive-dimensional normal subgroup of G. It follows then that $\mathfrak{h}$ contains no non-trivial ideal of $\mathfrak{g}$ and that the adjoint representation $\mathfrak{h}\to\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ of $\mathfrak{h}$ is faithful.

Now let H be a Lie group and ρ a representation of H on a vector space $\mathfrak{m}$. A vector X of $\mathfrak{m}$ is said to be *invariant by* H if we have $\rho(h)X=X$ for every $h\in H$. This being the case, we have $\tilde{\rho}(U)\cdot X=0$ for every $U\in\mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H and ρ is the representation of $\mathfrak{h}$ induced by ρ. In this case, X is called *invariant by* $\mathfrak{h}$.

LEMMA. *Let $\mathfrak{h}$ be a Lie algebra and ρ a representation of $\mathfrak{h}$ on a vector space $\mathfrak{m}$. Assume further that ρ is semi-simple. Then there is no non-zero vector of $\mathfrak{m}$ invariant by $\mathfrak{h}$ if and only if $\rho(\mathfrak{h})\cdot\mathfrak{m}=\mathfrak{m}$, where $\rho(\mathfrak{h})\cdot\mathfrak{m}$ is the space spanned by all elements of the form $\rho(U)\cdot X$, $U\in\mathfrak{h}$, $X\in\mathfrak{m}$.*

PROOF. Assuming that there is an invariant vector X_1 of $\mathfrak{m}$, let $\mathfrak{m}_1$ be the subspace spanned by X_1. Then we have $\rho(\mathfrak{h})\cdot\mathfrak{m}_1=0$. ρ being semi-simple, there exists an invariant subspace $\mathfrak{m}_2$ such that $\mathfrak{m}$ is the direct sum of $\mathfrak{m}_1$ and $\mathfrak{m}_2$, $\rho(\mathfrak{h})\cdot\mathfrak{m}_2\subset\mathfrak{m}_2$. Thus we have

$$\rho(\mathfrak{h})\cdot\mathfrak{m}=\rho(\mathfrak{h})\cdot(\mathfrak{m}_1+\mathfrak{m}_2)=\rho(\mathfrak{h})\cdot\mathfrak{m}_2\subset\mathfrak{m}_2\neq\mathfrak{m}.$$

Conversely, assume that $\rho(\mathfrak{h})\cdot\mathfrak{m}=\mathfrak{m}_2\neq\mathfrak{m}$, then the subspace $\mathfrak{m}_2$ is a proper invariant subspace of $\mathfrak{m}$ and there exists an invariant subspace $\mathfrak{m}_1$ such that $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2$ (direct). Then we have $\rho(\mathfrak{h})\cdot\mathfrak{m}_1\subset\mathfrak{m}_1\cap\mathfrak{m}_2=(0)$, which proves that $\mathfrak{m}$ has an invariant vector.

2. The property (A). The notation and assumptions being as

1) Cf. K. Nomizu: Invariant affine connections on homogeneous spaces, Amer. J. Math., **76**, 33-65 (1954).

in the preceding section, a homogeneous space G/H is said to have *the property* (A), if the following conditions are fulfilled:

(a) The representation $\mathfrak{h} \rightarrow \mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ is semi-simple.

(b) There is no non-zero vector of $\mathfrak{m}$ invariant by $\mathfrak{h}$.

(c) If $\mathfrak{m}_1$ is a subspace invariant by $\mathfrak{h}$ and $\mathfrak{m}_2$ the supplementary subspace of $\mathfrak{m}$ invariant by $\mathfrak{h}$, denoting by $\mathfrak{h}_2$ and $\mathfrak{h}_1$ the kernels of the representations $\mathfrak{h} \rightarrow \mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively, then we have $\mathfrak{h}=\mathfrak{h}_1+\mathfrak{h}_2$.

Using this terminology we shall prove the following theorem.

THEOREM 1. *If a reductive homogeneous space has the property* (A), *then its invariant affine connections are equivalent to the products of the invariant affine connections on two reductive homogeneous spaces.*

PROOF. The notation being as above, let G/H be the reductive homogeneous space in question. Setting $\mathfrak{g}_i=\mathfrak{m}_i+\mathfrak{h}_i$, $i=1,2$, we shall first prove that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}$, and $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

We shall determine the space $[\mathfrak{m}_1, \mathfrak{m}_2]$. Since the adjoint representation $\mathfrak{h} \rightarrow \mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ is faithful, the sum $\mathfrak{h}_1+\mathfrak{h}_2$ is direct and we have a decomposition $\mathfrak{g}=\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{h}_1+\mathfrak{h}_2$ of $\mathfrak{g}$ into the direct sum of vector spaces, where

$$[\mathfrak{h}_1, \mathfrak{m}_2]=[\mathfrak{h}_2, \mathfrak{m}_1]=[\mathfrak{h}_1, \mathfrak{h}_2]=(0).$$

Since $[\mathfrak{h}, \mathfrak{m}]=\mathfrak{m}$ by the above Lemma it follows that $[\mathfrak{h}_1, \mathfrak{m}_1]=\mathfrak{m}_1$ and $[\mathfrak{h}_2, \mathfrak{m}_2]=\mathfrak{m}_2$ and therefore

$$[\mathfrak{h}_i, [\mathfrak{m}_1, \mathfrak{m}_2]]=[\mathfrak{m}_1, \mathfrak{m}_2].$$

Let X be an element of $[\mathfrak{m}_1, \mathfrak{m}_2]$, then X is written as the form

$$X=X_1+X_2+U_1+U_2$$

where $X_i \in \mathfrak{m}_i$, $U_i \in \mathfrak{h}_i$, $i=1,2$. Then for any $V_i \in \mathfrak{h}_i$, we have

$$[V_i, X]=[V_i, X_i]+[V_i, U_i]$$

and therefore $[V_i, X] \in \mathfrak{m}_i+\mathfrak{h}_i$. It follows that

$$[\mathfrak{m}_1, \mathfrak{m}_2]=[\mathfrak{h}_i, [\mathfrak{m}_1, \mathfrak{m}_2]]\subset(\mathfrak{m}_1+\mathfrak{h}_1)\cap(\mathfrak{m}_2+\mathfrak{h}_2).$$

Hence we have $[\mathfrak{m}_1, \mathfrak{m}_2]=(0)$.

Next we shall consider the spaces $[\mathfrak{m}_i, \mathfrak{m}_i]$, $i=1,2$. $[\mathfrak{m}_1, \mathfrak{m}_1]$ is obviously invariant by $\mathrm{ad}(\mathfrak{h})$, that is to say, $[\mathfrak{h}, [\mathfrak{m}_1, \mathfrak{m}_1]]\subset[\mathfrak{m}_1, \mathfrak{m}_1]$ or more precisely

$$[\mathfrak{h}_1, [\mathfrak{m}_1, \mathfrak{m}_1]]\subset[\mathfrak{m}_1, \mathfrak{m}_1], \quad [\mathfrak{h}_2, [\mathfrak{m}_1, \mathfrak{m}_1]]=(0).$$

Let $X=X_1+X_2+U_1+U_2$ be an element of $[\mathfrak{m}_1, \mathfrak{m}_1]$, where $X_i \in \mathfrak{m}_i$, $U_i \in \mathfrak{h}_i$, $i=1,2$. Then for any $V_2 \in \mathfrak{h}_2$, we have $0=[V_2, X]=[V_2, X_2]+[V_2, U_2]$, where $[V_2, X_2] \in \mathfrak{m}_2$, and $[V_2, U_2] \in \mathfrak{h}_2$. This implies $X_2=0$ since $\mathfrak{m}_2$ has no non-zero vector invariant by $\mathfrak{h}_2$ in virtue of $[\mathfrak{h}_2, \mathfrak{m}_2]=\mathfrak{m}_2$. On the other hand, from $[\mathfrak{m}_1, \mathfrak{m}_2]=(0)$ it follows that for any $Y_2 \in \mathfrak{m}_2$ we have $[Y_2, X]=0$, hence we have $[Y_2, X]=[Y_2, U_2]=0$. This implies $U_2=0$ because of the effectiveness of G. Accordingly we have $X=X_1+U_2$ and $[\mathfrak{m}_1, \mathfrak{m}_1]\subset\mathfrak{m}_1+\mathfrak{h}_1=\mathfrak{g}_1$. In the same manner we have $[\mathfrak{m}_2, \mathfrak{m}_2]\subset\mathfrak{m}_2+\mathfrak{h}_2=\mathfrak{g}_2$.

These imply that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}$ and the sum $\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2$ is direct.

Next, let α be an arbitrary connection function[2] on G/H, i.e. a bilinear mapping of $\mathfrak{m}\times\mathfrak{m}$ into $\mathfrak{m}$ such that $\mathrm{ad}(h)\cdot\alpha(X, Y)=\alpha\,(\mathrm{ad}(h)\cdot X, \mathrm{ad}(h)\cdot Y)$ for all $X, Y\in\mathfrak{m}$, $h\in H$. Then, since $\mathfrak{m}_1$ is the set of elements $X\in\mathfrak{m}$ such that $[U, X]=0$ for all $U\in\mathfrak{h}_2$, it follows that for all $X_1, Y_1\in\mathfrak{m}_1$ we have $\alpha(X_1, Y_1)\in\mathfrak{m}_1$. In the same manner we have $\alpha(X_2, Y_2)\in\mathfrak{m}_2$ for all $X_2, Y_2\in\mathfrak{m}_2$. Now if for $X_1\in\mathfrak{m}_1$, $X_2\in\mathfrak{m}_2$ we set $\alpha(X_1, X_2)=Y_1+Y_2$, $Y_1\in\mathfrak{m}_1$, $Y_2\in\mathfrak{m}_2$, then for any $U_1\in\mathfrak{h}_1$ we have

$$[U_1, \alpha(X_1, X_2)]=\alpha([U_1, X_1], X_2).$$

On the other hand, we have

$$[U_1, \alpha(X_1, X_2)]=[U_1, Y_1]\in\mathfrak{m}_1.$$

Since $[\mathfrak{h}_1, \mathfrak{m}_1]=\mathfrak{m}_1$ it follows $\alpha(X_1, X_2)\in\mathfrak{m}_1$. In the same way we have $\alpha(X_1, X_2)\in\mathfrak{m}_2$. From $\mathfrak{m}_1\cap\mathfrak{m}_2=(0)$ it follows

$$\alpha(X_1, X_2)=\alpha(X_2, X_1)=0.$$

Thus we have proved that α maps $\mathfrak{m}_i\times\mathfrak{m}_i$ into $\mathfrak{m}_i$ and vanishes on $\mathfrak{m}_1\times\mathfrak{m}_2$. So we shall denote by α_i the restriction of α to $\mathfrak{m}_i\times\mathfrak{m}_i$. Let G_i and H_i be the subgroups of G generated by $\mathfrak{g}_i$ and $\mathfrak{h}_i$ respectively, then α_i is a connection function on the reductive homogeneous space G_i/H_i. It is clear that α is equivalent to the product of α_1 and α_2. Thus the proof of Theorem 1 is complete.

REMARK. It is easy to find an example showing the condition $[\mathfrak{h}, \mathfrak{m}]=\mathfrak{m}$ in Theorem 1 is really necessary.

We shall give some examples of homogeneous space having the property (A).

Let G/H be a homogeneous space of a Lie group G by a compact subgroup H. $\tilde{H}$ denotes the linear isotropy group at the point $p_0=H$ of G/H. Assume further that $\tilde{H}$ is a product of H_1 and H_2 and the elements of $\tilde{H}$ have the following type:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad A\in H_1, \qquad B\in H_2,$$

where H_1 and H_2 are irreducible groups of matrices. Under these assumptions G/H is reductive and has the property (A). Especially, if $\tilde{H}$ is the group of all matrices

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A\in R(k), \quad B\in R(n-k), \quad (k, n-k\geqq 2),$$

then the homogeneous Riemannian space G/H is locally isometric to a product of two Riemannian spaces of constant curvature of dimen-

2) See the foot-note 1).

sions k and $n-k$ respectively. Here $R(k)$ is the rotation group of a k-dimensional vector space.

3. **The property (B)**

Let M be an affinely connected manifold which is connected. If φ is an affine transformation leaving a point p of M fixed, $\tilde{\varphi}$ denotes the linear transformation on the tangent vector space Y_p at p induced by φ. Then the correspondence $\varphi \to \tilde{\varphi}$ is a faithful representation of the isotropy group at p, since φ is an affine transformation.

An affinely connected manifold M of n dimensions is called to have *the property* (B), if it is possible to find an affine transformation φ satisfying the following conditions:

(a) φ leaves some point p of M fixed,

(b) T_p has a base $\{X_1, \dots, X_n\}$ such that for some real numbers ρ_i

$$\tilde{\varphi} X_i = \rho_i X_i \ (1 \leqq i \leqq n),$$

(c) we have $\rho_i \rho_j \rho_k \rho_l^{-1} \neq 1$ if $j \neq k$ $(1 \leqq i, j, k, l \leqq n)$.

It is clear that ρ_i can not be zero.

Using this terminology we shall prove the following theorem.

THEOREM 2. *Let M be an affinely connected manifold admitting a transitive group of affine transformations. If M has the property (B), then the curvature tensor field R of M vanishes.*

PROOF. Let p be a point of M and φ an affine transformation by which the property (B) of M is defined. If R is the curvature tensor field of M and R_p the tensor at p, R_p is a multilinear mapping of $T_p \times T_p \times T_p \times T_p^*$ into the real numbers, where T_p^* is the dual of T_p and for any $X, Y, Z \in T_p$ and $\omega \in T_p^*$

$$R_p(X, Y, Z, \omega) = -R_p(X, Z, Y, \omega).$$

φ being an affine transformation, R_p is invariant by $\tilde{\varphi}$, i.e.

$$R_p(X, Y, Z, \omega) = R_p(\tilde{\varphi} X, \tilde{\varphi} Y, \tilde{\varphi} Z, \tilde{\varphi} \omega).$$

Now we take a base $\{X_1, \dots, X_n\}$ of T_p such that $\tilde{\varphi} X_i = \rho_i X_i$. If $\{\omega_1, \dots, \omega_n\}$ is the base of T_p^* defined by $\omega_i(X_j) = \delta_{ij}$, we have

$$\tilde{\varphi} \omega_i = \rho_i^{-1} \omega_i .$$

Then by the above equality we have

$$R_p(X_i, X_j, X_k, \omega_l) = \rho_i \rho_j \rho_k \rho_l^{-1} R_p(X_i, X_j, X_k, \omega_l).$$

If $j \neq k$, it follows $R_p(X_i, X_j, X_k, \omega_l) = 0$ by virtue of $\rho_i \rho_j \rho_k \rho_l^{-1} \neq 1$. If $j = k$ from the beginning $R_p(X_i, X_j, X_k, \omega_l) = 0$. This shows that R vanishes at p. Since M admits a transitive group of affine transformations, R vanishes at every point of M. Thus the proof is complete.

An affinely connected manifold M is called to have *the property* (B'), if there exists an affine transformation φ satisfying the conditions (a), (b), and

(c′) $\rho_i \rho_j \rho_k^{-1} \neq 1$ if $i \neq j$ $(1 \leqq i, j, k \leqq n)$.

Since the torsion tensor field T is invariant by the affine transformation, we have in the same manner as above the following theorem.

THEOREM 3. *Let M be an affinely connected manifold admitting a transitive group of affine transformations. If M has the property (B'), then the torsion tensor field T of M vanishes.*

It is easily seen that *if M admits a group of affine transformations of dimension $\geqq n^2+1$ $(\dim M=n)$, M has then the property (B) and (B') and the group is transitive. So M is locally flat, i.e. $R=0$, $T=0$.*