99. Groups of Isometries of Pseudo-Hermitian Spaces. II

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In the previous paper [4] we have given Theorem 4, in which the assumption $n \neq 3$, $n>1$ must be added. The notation and assumptions being as in the previous, we can state the following theorem including the case $n=3$.

THEOREM 4. *Let G/H be a homogeneous pseudo-Hermitian space of dimension $2n$ and $\dim G=n^2+2n-1$ $(n>1)$. If $n\neq 3$, G/H is flat and homeomorphic to E^{2n} and the group G is isomorphic to $S\mathfrak{M}_H(n)$. If $n=3$, G/H is flat or of positive constant curvature.*

In case $n=3$ and G/H is flat, the conclusion is the same as in the general case. In case $n=3$ and G/H is of positive constant curvature, G/H is homeomorphic to a sphere S^6 of dimension 6 and the group G is isomorphic to a compact exceptional simple group of type (G).

PROOF. Since H is isomorphic to $SU(n)$, there exists in the Lie algebra $\mathfrak{g}$ a subspace $\mathfrak{m}$ such that

$$\mathfrak{g}=\mathfrak{m}+\mathfrak{h} \qquad \text{(direct sum as vector space)},$$
$$[\mathfrak{h}, \mathfrak{m}]\subset \mathfrak{m},$$

where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to the subgroup H and $[\mathfrak{h}, \mathfrak{m}]$ denotes the subspace spanned by all elements of the form $[U, X]$, $U\in\mathfrak{h}$, $X\in\mathfrak{m}$.

First, we have easily

$$[\mathfrak{h}, [\mathfrak{m}, \mathfrak{m}]]\subset [\mathfrak{m}, \mathfrak{m}],$$

where $[\mathfrak{m}, \mathfrak{m}]$ denotes the subspace spanned by all elements of the form $[X, Y]$, $X, Y\in\mathfrak{m}$. Since $\mathfrak{h}$ is simple, one of the following four cases occurs:

$$[\mathfrak{m}, \mathfrak{m}]=\{0\}, \quad [\mathfrak{m}, \mathfrak{m}]=\mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}]=\mathfrak{h}, \quad [\mathfrak{m}, \mathfrak{m}]=\mathfrak{g}.$$

The first three cases have been discussed in the previous paper. When the last case occurs, it is easily seen that $\mathfrak{g}$ is simple and $\dim\mathfrak{g}=n^2+2n-1$. Looking over the table of simple Lie algebras due to É. Cartan [2, p. 49], we see that, if there exists such a simple Lie algebra $\mathfrak{g}$, n must be 3 and $\mathfrak{g}$ the exceptional simple Lie algebra $\mathfrak{g}_{14}$ of type (G) which is of rank 2 and of dimension 14. Thus Theorem 4 is proved for $n\neq 3$.

To prove Theorem 4 for $n=3$, assume that $\mathfrak{g}$ has the structure of the exceptional simple Lie algebra of type (G). It has been proved by É. Cartan [3, pp. 292-298] that there exist exactly two real forms of simple Lie algebra $\mathfrak{g}_{14}$, one of them, say $\mathfrak{g}_1$, is compact and the other, say $\mathfrak{g}_2$, is non-compact.

We shall now prove $\mathfrak{g} \neq \mathfrak{g}_2$. It is sufficient to show that $\mathfrak{g}_2$ does not contain the Lie algebra of $SU(3)$ as a subalgebra. For this purpose it is useful for us to recall a result of É. Cartan [3, pp. 292–298] that there is a linear group G_2: $E^7 \to E^7$ leaving invariant a quadratic form Q of signature 3 and seven alternating bilinear forms where G_2 is a Lie group generated by $\mathfrak{g}_2$. For brevity, we shall assume for G_2 to be connected.

Let $\mathfrak{L}(7)$ be the identity component of the group of all linear transformations on E^7 leaving the quadratic form Q invariant. It is evident that $\mathfrak{L}(7) \supset G_2$. It is not hard to see that the maximal compact subgroups of $\mathfrak{L}(7)$ are isomorphic to the product group $R(4) \times R(3)$, where $R(4)$ and $R(3)$ are the rotation groups of E^4 and E^3 respectively.

If we assume that $\mathfrak{g} = \mathfrak{g}_2$, then $\mathfrak{h}$ is contained in $\mathfrak{g}_2$ and consequently $\mathfrak{h}$ is contained in the Lie algebra of $\mathfrak{L}(7)$. $\mathfrak{h}$ being a compact simple Lie algebra, we can assume without loss of generality $\mathfrak{h} \subset \mathfrak{r}(4) \times \mathfrak{r}(3)$, where $\mathfrak{r}(4)$ and $\mathfrak{r}(3)$ are the Lie algebras of $R(4)$ and $R(3)$ respectively. But this contradicts that $\mathfrak{h}$ is simple and $\dim \mathfrak{h} = 8$. Thus we have $\mathfrak{g} \neq \mathfrak{g}_2$.

Next we shall consider the case $\mathfrak{g} = \mathfrak{g}_1$. É. Cartan [3, pp. 292–298] has showed that there is a linear group G_1 on E^7 leaving invariant a positive definite quadratic form and seven alternating bilinear forms where G_1 is a Lie group generated by $\mathfrak{g}_1$. Moreover, it is well known that G_1 is the group of all automorphisms of the Cayley algebra [5, pp. 212–216]. By virtue of this fact, the unit sphere S^6 in E^7 is representable as a homogeneous space G_1/H_1, where H_1 is isomorphic to $SU(3)$, and the homogeneous space $S^6 = G_1/H_1$ is of positive constant curvature as a homogeneous Riemannian space.

Since every Lie group having $\mathfrak{g}_1$ as its Lie algebra is simply connected [6, § 5], we can identify two groups G and G_1. On the other hand two simple subgroups H and H_1 of $G = G_1$ have the same rank as G_1 and they are isomorphic to each other. From the table given by A. Borel [1, § 7], especially in this case, the following fact holds true: There exists in $\mathfrak{g}_1$ one and only one compact simple subalgebra $\mathfrak{w}$ containing an arbitrary given maximal Abelian subalgebra of $\mathfrak{g}_1$, if $\mathfrak{w}$ is of rank 2 and of dimension 8. Then two subalgebras $\mathfrak{h}$ and $\mathfrak{h}_1$ are conjugate to each other in $\mathfrak{g}_1$, where $\mathfrak{h}_1$ is the subalgebra of $\mathfrak{g}_1$ corresponding to H_1. Thus two subgroups H and H_1 are conjugate in $G = G_1$ to each other, because G_1 is simply connected and H and H_1 are connected. Consequently, the given homogeneous space G/H can be identified with G_1/H_1. This completes the proof of Theorem 4.

In Theorem 4, *if H is not connected, G/H is homeomorphic to the real projective space of dimension* 6 *and has positive constant curvature* [7, § 5].

At the last of the previous paper [4] the following proposition was given: *If G/H is a homogeneous space of dimension $4n$ and H is isomorphic to $Sp(n)$, then G/H is locally flat and homeomorphic to E^{4n}.* In quite analogous manner to the proof of Theorem 4, we can see that there is no exceptional case in the above proposition.

Theorem 3 in the previous paper has to be corrected as follows.

THEOREM 3. *Let G/H be a homogeneous pseudo-Hermitian space of dimension $2n$ and $\dim G=n^2+2n$. Then G/H is a homogeneous pseudo-Kählerian space with constant holomorphic sectional curvature K. When $K>0$ and G/H is simply connected, G is isomorphic locally to $SU(n+1)$ and G/H is homeomorphic to $P(C,n)$. When $K<0$, G is locally isomorphic to $S\mathfrak{L}(n+1)$ and G/H is homeomorphic to E^{4n}. When $K=0$, G is isomorphic to $\mathfrak{M}_H(2n)$ and G/H is homeomorphic to E^{4n}.*

References

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