94. Some Trigonometrical Series. XV

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1. G. H. Hardy and J. E. Littlewood [1] have proved the following

Theorem 1. *Let $p\geqq 1$, $0<\alpha<1$, and $\alpha p>1$. If $f(x)$ belongs to the Lip (α, p) class, then $f(x)$ is equivalent to a function belonging to the Lip $(\alpha-1/p)$ class.*

This was generalized in the following form [2]:

Theorem 2. *Let $p\geqq 1$, $0<\alpha<1$, and $\alpha p>1$. If $f(x)$ belongs to the Lip (α, p) class, then*

$$f(x)-s_n(x)=O(1/n^{\alpha-1/p}) \tag{1}$$

uniformly almost everywhere, where $s_n(x)$ is the nth partial sum of the Fourier series of $f(x)$.

If is well known that (1) implies that $f(x)$ is equivalent to a function of the Lip $(\alpha-1/p)$ class.

In the proof of Theorem 2 in [2], Theorem 1 is used. We shall prove here Theorem 2, without using Theorem 1, but using the idea in [1]. From this proof we get the following

Theorem 3. *Let $p\geqq 1$, $0<\alpha<1$, and $\alpha p<1$. If $f(x)$ belongs to the Lip (α, p) class, then*

$$s_n(x)-f(x)=O(n^{1/p-\alpha}) \tag{2}$$

uniformly almost everywhere.

It is known that if $f(x)$ belongs to the L^p class $(p\geqq 1)$, then

$$s_n(x)=o(n^{1/p})$$

and the exponent $1/p$ is the best possible one. Estimation of the integral mean of the left side of (2) was given by E. S. Quade [3].

2. We shall prove Theorem 2. Let $f(x)$ be a function in the Lip (α, p) class, and let

$$\varphi_x(t)=f(x+t)+f(x-1)-2f(x)$$

and $s_n(x)$ be the nth partial sum of the Fourier series of $f(t)$ at $t=x$. Then (cf. [4])

$$|s_n(x)-f(x)|\leqq \int_{\pi/n}^{\pi}\frac{\varphi_x(t)-\varphi_x(t+\pi/n)}{t^2}dt + \frac{A}{n}\int_{\pi/n}^{\pi}\frac{|\varphi_x(t)|}{t^2}dt+2n\int_0^{2\pi/n}|\varphi_x(t)|dt+O(1/n^\alpha).$$

If we prove that

$$\int_0^t |\varphi_x(u)|\,du = O(t^{1+\alpha-1/p}) \tag{3}$$

and

$$\int_t^\pi \frac{|\varphi_x(t+u)-\varphi_x(u)|}{u}\,du = O(t^{\alpha-1/p}) \tag{4}$$

uniformly almost everywhere, then we get (1) and hence $f(x)$ belongs to the $\mathrm{Lip}\,(\alpha-1/p)$ class. Thus we get Theorems 1 and 2.

Proof of (4) is easy. For, putting $1/p+1/q=1$,

$$\int_t^\pi |\varphi_x(u+t)-\varphi_x(u)|\frac{du}{u} \leqq \Big(\int_t^\pi |\varphi_x(u+t)-\varphi_x(u)|^p\,du\Big)^{1/p}\Big(\int_t^\pi u^{-q}\,du\Big)^{1/q}$$

$$\leqq 2\Big(\int_{-\pi}^\pi |f(u+t)-f(u)|^p\,du\Big)^{1/p} t^{-(q-1)/q} \leqq At^{\alpha-1/p}.$$

3. It remains to prove (3), that is,

Lemma. *Let $p\geqq 1$, $0<a<1$, and $\alpha p>1$. If $f(x)$ belongs to the Lip (α, p) class, then*

$$\int_0^h |f(x+u)-f(x)|\,du = O(h^{1+\alpha-1/p}) \tag{5}$$

uniformly almost everywhere.

This is weaker than Theorem 1.

In order to prove this lemma, we can suppose that $f(x)$ is of power series type, since the conjugate function of $f(x)$ belongs also to the $\mathrm{Lip}\,(\alpha, p)$ class [1]. Let $F(z)$ be the regular function in the unit circle with $f(x)$ as the boundary function. Then, by the Cauchy theorem,

$$F'(re^{ix}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(t)\,e^{it}}{(e^{it}-re^{ix})^2}\,dt = \frac{e^{-ix}}{2\pi}\int_0^{2\pi}\frac{f(x+t)\,e^{it}}{(e^{it}-r)^2}\,dt$$

$$= \frac{e^{-ix}}{2\pi}\int_0^{2\pi}\frac{e^{it}}{(e^{it}-r)^2}(f(x+t)-f(x))\,dt.$$

By the assumption (cf. [1])

$$\Big(\int_0^{2\pi} |F'(re^{ix})|^p\,dx\Big)^{1/p} \leqq \frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|e^{it}-r|^2}\Big(\int_0^{2\pi}|f(x+t)-f(x)|^p\,dx\Big)^{1/p} \tag{6}$$

$$\leqq A\int_0^\pi \frac{t^\alpha}{|e^{it}-r|^2}\,dt \leqq A\int_0^\infty \frac{t^\alpha}{(1-r)^2+t^2}\,dt \leqq A(1-r)^{-1+\alpha}.$$

In order to prove (5), it suffices to prove that

$$\int_0^h |F(re^{i(x+u)})-F(re^{ix})|\,du \leqq Ah^{1+\alpha-1/p} \tag{7}$$

uniformly for all $r<1$. Let C_0 and C_2 be the circular arcs with radius r and $r-h$, respectively and with end points of arguments x and $x+u$; C_1 and C_3 be the segments:

$$C_1 = [(r-h)e^{i(x+u)}, re^{i(x+u)}], \quad C_3 = [(r-h)e^{ix}, re^{ix}].$$

Then the left side of (6) is

$$\int_0^h du \left| \int_{C_0} F'(z)\, dz \right| \leqq \sum_{i=1}^{3} \int_0^h du \left| \int_{C_i} F'(z)\, dz \right| = \sum_{i=1}^{3} I_i .$$

Now, taking r sufficiently near 1,

$$I_1 \leqq \int_{r-h}^{h} d\rho h^{1/q} \left(\int_0^{2\pi} | F'(\rho e^{iu}) |^p du \right)^{1/p}$$
$$\leqq A h^{1/q} \int_{r-h}^{r} \frac{d\rho}{(1-\rho)^{1-\alpha}} \leqq A h^{1+\alpha-1/p}$$

and

$$I_2 \leqq \int_0^h du \int_x^{x-u} | F'((r-h)e^{iv}) | \, dv$$
$$\leqq \int_0^h u^{1/q} du \left(\int_0^{2\pi} | F'((r-h)e^{iv}) |^p dv \right)^{1/p} \leqq A h^{1+\alpha-1/p}$$

Finally, by (6),

$$F'(re^{ix}) = \left| \frac{\rho}{2\pi} \int_{-\pi}^{\pi} \frac{F'(\rho e^{iu}) e^{iu}}{\rho e^{iu} - r e^{ix}} du \right|$$
$$\leqq A \left(\int_{-\pi}^{\pi} | F'(\rho e^{iu}) |^p du \right)^{1/p} \left(\int_{-\pi}^{\pi} \frac{du}{| \rho e^{iu} - r e^{ir} |^q} \right)^{1/q}$$
$$\leqq A (1-\rho)^{-1+\alpha} \left(\int_0^{\infty} \frac{du}{((\rho - r)^2 + u^2)^{q/2}} \right)^{1/q}$$
$$\leqq A (1-\rho)^{-1+\alpha} (\rho - r)^{-1/p} \leqq A (1-r)^{-1+\alpha-1/p},$$

taking $\rho = (1-r)/2$ (cf. [1]). Hence

$$I_3 \leqq h \int_{r-h}^{r} | F'(\rho e^{ix}) | \, d\rho$$
$$\leqq A h \int_{r-h}^{r} (1-\rho)^{-1+\alpha-1/p} \, d\rho \leqq A h^{1+\alpha-1/p}$$

4. Proof of Theorem 3 is easy from that of Theorem 2.

References

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